MATH 617 (WINTER 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 1

This homework assignment is due Wednesday 17 January 2024.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Problem 1 (Problem A)**.** Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, let $h \colon X \to Y$ be a measurable function (that is, if $F \in \mathcal{N}$ then $h^{-1}(F) \in \mathcal{M}$), and let μ be a measure on $(X, \mathcal{M})$. Define $\nu \colon \mathcal{N} \to [0, \infty]$ by $\nu(F) = \mu(h^{-1}(F))$ for $F \in \mathcal{N}$. Prove the following:

(1) ν is a measure on $(X, \mathcal{M})$.
(2) If $f \colon Y \to [0, \infty]$ is measurable, then $f \circ h \colon X \to [0, \infty]$ is measurable, and $\int_X (f \circ h) \, d\mu = \int_Y f \, d\nu$.
(3) If $f \colon Y \to \mathbb{C}$ is integrable, then $f \circ h \colon X \to \mathbb{C}$ is integrable, and $\int_X (f \circ h) \, d\mu = \int_Y f \, d\nu$.

The measure ν is called the push forward of μ under h, and written $h_*(\mu)$.

Solution. For Part (1), it is trivial that $\nu(\varnothing) = 0$. If $F_1, F_2, \ldots \in \mathcal{N}$ are disjoint, then also $h^{-1}(F_1)$, $h^{-1}(F_2)$, $\ldots$ are disjoint. Since inverse images preserve unions, we thus get

$$\nu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(\bigcup_{n=1}^{\infty} h^{-1}(F_n)\right) = \sum_{n=1}^{\infty} \mu(h^{-1}(F_n)) = \sum_{n=1}^{\infty} \nu(F_n).$$

This proves that ν is a measure.

We prove Part (2). Measurability of $f \circ h$ is Theorem 1.7(b) in Rudin's book.

Next, let $F \subset Y$ be measurable. Then $\chi_F \circ h = \chi_{h^{-1}(F)}$, so

$$\int_X (\chi_F \circ h) \, d\mu = \mu(h^{-1}(F)) = \nu(F) = \int_Y \chi_F \, d\nu.$$

This proves that

$$\int_X (f \circ h) \, d\mu = \int_Y f \, d\nu \tag{1}$$

whenever f is the characteristic function of a measurable set. It is now immediate that (1) holds whenever f is a nonnegative simple function. Now let $f \colon Y \to [0, \infty]$ be measurable. Choose a sequence $(f_n)_{n \in \mathbb{Z}_{>0}}$ of nonnegative simple functions on Y which increases pointwise to f. Then $(f_n \circ h)_{n \in \mathbb{Z}_{>0}}$ increases pointwise to $f \circ h$. Using the Monotone Convergence Theorem at the first and third steps, and (1) for nonnegative simple functions at the second step, we then have

$$\int_X (f \circ h) \, d\mu = \lim_{n \to \infty} \int_X (f_n \circ h) \, d\mu = \lim_{n \to \infty} \int_Y f_n \, d\nu = \int_Y f \, d\nu.$$

This completes the proof of Part (2).

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We prove (3). Again, measurability of $f \circ h$ is Theorem 1.7(b) in Rudin's book. The integrals

$$\int_X |f \circ h| \, d\mu \qquad \text{and} \qquad \int_Y |f| \, d\nu$$

are equal by Part (2), so are either both finite or both infinite. This says that $f \circ h$ is integrable if and only if f is integrable.

When f is integrable, so are $\mathrm{Re}(f)_+$, $\mathrm{Re}(f)_-$, $\mathrm{Im}(f)_+$, and $\mathrm{Im}(f)_-$. Thus, using Part (2),

$$\begin{aligned}
\int_X (f \circ h) \, d\mu &= \int_X [\mathrm{Re}(f)_+ \circ h] \, d\mu - \int_X [\mathrm{Re}(f)_- \circ h] \, d\mu \\
&\qquad + i \int_X [\mathrm{Im}(f)_+ \circ h] \, d\mu - i \int_X [\mathrm{Im}(f)_- \circ h] \, d\mu \\
&= \int_X \mathrm{Re}(f)_+ \, d\nu - \int_X \mathrm{Re}(f)_- \, d\nu + i \int_X \mathrm{Im}(f)_+ \, d\nu - i \int_X \mathrm{Im}(f)_- \, d\nu \\
&= \int_Y f \, d\nu.
\end{aligned}$$

This completes the solution. $\square$

Problem 2 (Problem 13 in Chapter 4 of Rudin)**.** Let $f \colon \mathbb{R} \to \mathbb{C}$ be a 1-periodic continuous function. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t) \, dt.$$

Hint. First consider the case $f(t) = e^{2\pi i k t}$ for some $k \in \mathbb{Z}$.

Proof. We first follow the hint.

Let $k \in \mathbb{Z}$, and set $f(t) = e^{2\pi i k t}$ for $t \in \mathbb{R}$. First suppose $k = 0$. Then

$$\frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \frac{1}{N} \sum_{n=1}^{N} 1 = 1 = \int_0^1 f(t) \, dt.$$

Now suppose $k \neq 0$. Then $e^{2\pi i k \alpha} \neq 1$. For $N \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}
\left| \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) \right| &= \frac{1}{N} \left| \sum_{n=1}^{N} \left(e^{2\pi i k \alpha} \right)^n \right| = \frac{|e^{2\pi i k \alpha}|}{N} \left| \frac{1 - \left(e^{2\pi i k \alpha} \right)^n}{1 - e^{2\pi i k \alpha}} \right| \\
&\leq \frac{1}{N} \left(\frac{2}{|1 - e^{2\pi i k \alpha}|} \right).
\end{aligned}$$

The second factor is constant, so

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = 0 = \int_0^1 f(t) \, dt.$$

This completes the verification of the statement in the hint.

It immediately follows by linearity that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(n\alpha) = \int_0^1 g(t) \, dt \tag{2}$$

whenever f is a finite linear combination of the functions in the hint.

Let E be the vector space of 1-periodic continuous functions g on $\mathbb{R}$ with the norm $\|g\|_\infty = \sup_{t\in\mathbb{R}} |g(t)|$. For the same reasons as used in class for the identification of the 2π-periodic continuous functions on $\mathbb{R}$ with $C(S^1)$, the map which sends $f \in C(S^1)$ to the function $t \mapsto f(e^{2\pi i t})$ is an isometric bijection from $C(S^1)$ to E. Since the trigonometric polynomials are dense in $C(S^1)$ in $\|\cdot\|_\infty$, it follows that linear combination of the functions in the hint are dense in E.

Now let $f \in E$ and let $\varepsilon > 0$. Choose m and $\lambda_{-m}, \lambda_{-m+1}, \ldots, \lambda_{m-1}, \lambda_m$ such that the function

$$g(t) = \sum_{k=-m}^{m} \lambda_k e^{2\pi i k t}$$

satisfies $\|g - f\|_\infty < \frac{\varepsilon}{3}$. Then, using (2) for g at the first step,

$$\begin{aligned}
&\left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) - \int_0^1 f(t)\,dt \right| \\
&\qquad \le \left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(n\alpha) \right| + \left| \int_0^1 g(t)\,dt - \int_0^1 f(t)\,dt \right| \\
&\qquad \le \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |f(n\alpha) - g(n\alpha)| + \int_0^1 |g(t) - f(t)|\,dt \\
&\qquad \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the desired conclusion follows. □

Remark 1. In the last paragraph of the proof, taking $\|g - f\|_\infty < \frac{\varepsilon}{2}$ will work. But, then, to get "$< \varepsilon$" at the end, one needs to argue that $\int_0^1 |g(t) - f(t)|\,dt < \frac{\varepsilon}{2}$. This is true, but more annoying than requiring $\|g - f\|_\infty < \frac{\varepsilon}{3}$ to begin with.

Problem 3 (Problem 4 in Chapter 5 of Rudin)**.** Define $M \subset C([0,1])$ by

$$M = \left\{ f \in C([0,1]) \colon \int_0^{1/2} f(t)\,dt - \int_{1/2}^{1} f(t)\,dt = 1 \right\}.$$

Prove that M is a closed convex set which contains no element of minimal norm.

Why does this not contradict Theorem 4.10 of Rudin's book?

Proof. Define $\omega \colon C([0,1]) \to \mathbb{C}$ by

$$\omega(f) = \int_0^{1/2} f(t)\,dt - \int_{1/2}^{1} f(t)\,dt$$

for $f \in C([0,1])$. Then ω is clearly a linear functional. For $f \in C([0,1])$, we have

$$|\omega(f)| \le \int_0^{1/2} |f(t)|\,dt + \int_{1/2}^{1} |f(t)|\,dt = \int_0^1 |f(t)|\,dt \le \|f\|_\infty.$$

This shows that ω is continuous, with $\|\omega\| \le 1$. Therefore $M = \omega^{-1}(\{1\})$ is closed. Also, since $\{1\}$ is convex and ω is linear, it is immediate that $M = \omega^{-1}(\{1\})$ is convex.

We claim that $\inf_{f \in M} \|f\|_\infty = 1$. To prove the claim, suppose $\|f\|_\infty < 1$. Then, just because $\|\omega\| \leq 1$, we must have $|\omega(f)| < 1$. Thus $\inf_{f \in M} \|f\|_\infty \geq 1$. To prove the reverse inequality, let $\varepsilon > 0$, and assume that $\varepsilon < 2$. Define

$$\lambda = 1 + \frac{\varepsilon}{2} \qquad \text{and} \qquad \delta = 1 - \frac{1}{1 + \frac{\varepsilon}{2}}.$$

Then define $f \in C([0,1])$ by

$$f(t) = \begin{cases} \lambda & 0 \leq t \leq \frac{1}{2} - \delta \\ \lambda\delta^{-1}\left(\frac{1}{2} - t\right) & \frac{1}{2} - \delta < t < \frac{1}{2} + \delta \\ -\lambda & \frac{1}{2} + \delta \leq t \leq 1. \end{cases}$$

Then one can check that $\|f\|_\infty = \lambda < 1 + \varepsilon$ and $\omega(f) = 1$. Since $\varepsilon \in (0,2)$ is arbitrary, the claim follows.

We now claim that there is no real valued $f \in M$ with $\|f\| \leq 1$. So suppose $\|f\| \leq 1$; we prve $f \notin M$. First suppose $f\left(\frac{1}{2}\right) \leq 0$. Choose $\delta > 0$ such that $f(t) < \frac{1}{2}$ whenever $\frac{1}{2} - \delta < t \leq \frac{1}{2}$. Then

$$\int_0^{1/2} f(t)\,dt = \int_0^{1/2-\delta/2} f(t)\,dt + \int_{1/2-\delta/2}^{1/2} f(t)\,dt \leq \frac{1}{2} - \frac{\delta}{2} + \frac{1}{2}\left(\frac{\delta}{2}\right) = \frac{1}{2} - \frac{\delta}{4},$$

while

$$\int_{1/2}^1 f(t)\,dt \geq -\frac{1}{2},$$

so $\omega(f) \leq 1 - \frac{\delta}{4}$. Thus $f \notin M$.

Now suppose instead that $f\left(\frac{1}{2}\right) \geq 0$. Define $g \in C([0,1])$ by $g(t) = -f(-t)$. One readily checks that also $\|g\|_\infty = 1$ and $\omega(g) = \omega(f)$. The previous paragraph shows that $\omega(g) < 1$, so also $\omega(f) < 1$. Thus $f \notin M$. The claim is proved.

Now let $f \in C([0,1])$ satisfy $\|f\| = 1$. Set $g = \mathrm{Re}(f)$. One immediately checks that $\omega(g) = \mathrm{Re}(\omega(f))$ and $\|g\| \leq 1$. The claim implies $\omega(g) < 1$. Therefore $\mathrm{Re}(\omega(f)) < 1$, so $f \notin M$. We have proved that M contains no elements with norm 1, proving the main statement.

For the second part, since $C([0,1])$ is complete, Theorem 4.10 of Rudin's book and the existence of M show that the standard norm on $C([0,1])$ does not come from a scalar product in the way that the norm on a Hilbert space does. □

Remark 2. Much more is actually true, although we might not see it in this course: there is no Hilbert space norm on $C([0,1])$ which even defines the same topology as $\|\cdot\|_\infty$.

Problem 4 (Problem 5 in Chapter 5 of Rudin)**.** Define $M \subset L^1([0,1])$ (using Lebesgue measure) by

$$M = \left\{ \xi \in L^1([0,1]) \colon \int_0^1 \xi(t)\,dt = 1 \right\}.$$

Prove that M is a closed convex set which contains infinitely many element of minimal norm.

Why does this not contradict Theorem 4.10 of Rudin's book?

Solution. Define $\omega \colon L^1([0,1]) \to \mathbb{C}$ by $\omega(\xi) = \int_0^1 \xi(t)\,dt$ for $\xi \in L^1([0,1])$. Then ω is clearly a linear functional. For $\xi \in L^1([0,1])$, we have $|\omega(\xi)| \leq \int_0^1 |\xi(t)|\,dt = \|\xi\|_1$.

This shows that ω is continuous. Therefore $M = \omega^{-1}(\{1\})$ is closed. Also, since $\{1\}$ is convex and ω is linear, it is immediate that $M = \omega^{-1}(\{1\})$ is convex.

If $\int_0^1 \xi(t)\,dt = 1$, then $\|\xi\|_1 = \int_0^1 |\xi(t)|\,dt \geq 1$. Therefore all elements of M have norm at least 1.

For $\alpha \in \left[0, \frac{1}{2}\right]$, set $\xi_\alpha = 2\chi_{[\alpha,\, \alpha+1/2]}$, twice the characteristic function of $\left[\alpha,\, \alpha + \frac{1}{2}\right]$. The functions ξ_α are distinct elements of $L^1([0,1])$, since if $\alpha \neq \beta$ then it is not true that $\xi_\alpha = \xi_\beta$ almost everywhere. Clearly $\xi_\alpha \in M$ and $\|\xi_\alpha\| = 1$ for all $\alpha \in \left[0, \frac{1}{2}\right]$. This proves the main statement.

For the second part, since $L^1([0,1])$ is complete, Theorem 4.10 of Rudin's book and the existence of M show that the standard norm on $L^1([0,1])$ does not come from a scalar product in the way that the norm on a Hilbert space does. □

Remark 3. Much more is actually true, although we might not see it in this course: there is no Hilbert space norm on $L^1([0,1])$ which even defines the same topology as $\|\cdot\|_1$.

Problem 5 (Problem 14 in Chapter 5 of Rudin)**.** For $n \in \mathbb{Z}_{>0}$ define a subset $X_n \subset C([0,1])$ by

$$X_n = \big\{ f \in C([0,1]) \colon \text{there is } t \in [0,1] \text{ such that } |f(s-t)| \leq n|s-t| \text{ for all } s \in [0,1] \big\}.$$

(1) Let $n \in \mathbb{Z}_{>0}$ and let $U \subset C([0,1])$ be a nonempty open set. Prove that there is a nonempty open set $V \subset U$ such that $V \cap X_n = \varnothing$.
(2) Prove that there is a dense G_δ-set $G \subset C([0,1])$ such that every function $f \in G$ is nowhere differentiable.

Hint. For (1), every function $f \in C([0,1])$ can be uniformly approximated by a zigzag function g with very large slopes. If these slopes are large enough, and $\|g - h\|_\infty$ is small enough, then $h \notin X_n$.

It is convenient to introduce some notation and prove a lemma.

Notation 4. Let X be any metric space, with metric ρ. For $x \in X$ and $r > 0$ we let $B_r(x)$ denote the open ball about x of radius r, that is,

$$B_r(x) = \big\{ y \in X \colon \rho(y, x) < r \big\}.$$

If X must be specified, we write $B_r^X(x)$.

Notation 5. Let $m \in \mathbb{Z}_{>0}$, suppose $0 = t_0 < t_1 < \cdots < t_m = 1$, and let $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$. Let $f_{m, t_0, t_1, \ldots, t_m, \lambda_0, \lambda_1, \ldots, \lambda_m}$ be the continuous piecewise linear function with nodes $t_0, t_1, \ldots, t_m$ and values $\lambda_0, \lambda_1, \ldots, \lambda_m$ at these nodes. That is, for $j = 1, 2, \ldots, m$ and $t \in [t_{j-1}, t_j]$,

$$f_{m, t_0, t_1, \ldots, t_m, \lambda_0, \lambda_1, \ldots, \lambda_m}(t) = \lambda_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}(\lambda_j - \lambda_{j-1}). \tag{3}$$

Lemma 6. Let $m \in \mathbb{Z}_{>0}$, suppose $0 = t_0 < t_1 < \cdots < t_m = 1$, and let $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$. Let $\alpha \in [0, \infty)$ satisfy

$$\alpha < \min_{1 \leq j \leq m} \frac{|\lambda_j - \lambda_{j-1}|}{t_j - t_{j-1}}.$$

Then there exists $\delta > 0$ such that whenever $g\colon [0,1] \to \mathbb{C}$ satisfies

$$\sup_{t\in[0,1]} \left| g(t) - f_{m,t_0,t_1,\ldots,t_m,\lambda_0,\lambda_1,\ldots,\lambda_m}(t) \right| < \delta,$$

then for every $t \in [0,1]$ there is $s \in [0,1]$ satisfying $|g(s) - g(t)| > \alpha|s - t|$.

Proof. Set $f = f_{m,t_0,t_1,\ldots,t_m,\lambda_0,\lambda_1,\ldots,\lambda_m}$. Define

$$\rho = \frac{1}{2}\min_{1\le j\le m}(t_j - t_{j-1}), \qquad \sigma = \min_{1\le j\le m}\frac{|\lambda_j - \lambda_{j-1}|}{t_j - t_{j-1}} - \alpha, \qquad \text{and} \qquad \delta = \frac{\sigma\rho}{2}.$$

Suppose $\sup_{t\in[0,1]}\left|g(t) - f(t)\right| < \delta$, and let $t \in [0,1]$. Choose $j \in \{1, 2, \ldots, m\}$ such that $t \in [t_{j-1}, t_j]$. We must have $t - t_{j-1} \geq \rho$ or $t_j - t \geq \rho$.

In the first case, set $s = t_{j-1}$. We have

$$\begin{aligned} |g(s) - g(t)| > |f(s) - f(t)| - 2\delta &= (t-s)\frac{|\lambda_j - \lambda_{j-1}|}{t_j - t_{j-1}} - \rho\sigma \\ &\geq (t-s)(\sigma+\alpha) - \rho\sigma = (t-s)\alpha + (t-s-\rho)\sigma \\ &\geq (t-s)\alpha = \alpha|t-s|. \end{aligned}$$

In the second case, set $s = t_j$. We have

$$\begin{aligned} |g(s) - g(t)| > |f(s) - f(t)| - 2\delta &= \left|\lambda_j - \lambda_{j-1} - (t - t_{j-1})\frac{|\lambda_j - \lambda_{j-1}|}{t_j - t_{j-1}}\right| - 2\delta \\ &= |t_j - t|\frac{|\lambda_j - \lambda_{j-1}|}{t_j - t_{j-1}} - 2\delta \geq |t_j - t|(\sigma+\alpha) - 2\delta \\ &\geq |t_j - t|\alpha + \rho\sigma - 2\delta = \alpha|s-t|. \end{aligned}$$

This completes the proof. □

Proof of Part (1). Choose any element $f \in U$. Choose $\varepsilon > 0$ such that $B_\varepsilon(f) \subset U$. By uniform continuity, there is $\delta > 0$ such that for all $s, t \in [0,1]$ with $|s - t| < \delta$, we have $|f(s) - f(t)| < \frac{\varepsilon}{9}$. Choose $m \in \mathbb{Z}_{>0}$ so large that

$$\frac{1}{m} < \min\left(\delta, \frac{\varepsilon}{9n}\right).$$

Set

$$\alpha = \frac{n}{m} + \frac{2\varepsilon}{9}.$$

For $j = 0, 1, \ldots, m$ define

$$t_j = \frac{j}{m} \qquad \text{and} \qquad \lambda_j = \begin{cases} f(t_j) & j \text{ is even} \\ f(t_j) + \alpha & j \text{ is odd.} \end{cases}$$

Set $g = f_{m,t_0,t_1,\ldots,t_m,\lambda_0,\lambda_1,\ldots,\lambda_m}$.

We claim that for $j = 1, 2, \ldots, m$, we have

$$m\left(\alpha - \frac{\varepsilon}{9}\right) < \frac{|\lambda_j - \lambda_{j-1}|}{t_j - t_{j-1}} < m\left(\alpha + \frac{\varepsilon}{9}\right), \tag{4}$$

and that $\|g - f\|_\infty \leq \frac{8\varepsilon}{9}$.

For the first part of the claim, regardless of whether j is even or odd, we have

$$\alpha - |f(t_j) - f(t_{j-1})| \leq |\lambda_j - \lambda_{j-1}| \leq \alpha + |f(t_j) - f(t_{j-1})|.$$

Since $t_j - t_{j-1} = \frac{1}{m} < \delta$, we have $|f(t_j) - f(t_{j-1})| < \frac{\varepsilon}{9}$, giving

$$\alpha - \frac{\varepsilon}{9} < |\lambda_j - \lambda_{j-1}| < \alpha + \frac{\varepsilon}{9}.$$

Diving by $t_j - t_{j-1} = \frac{1}{m}$ gives (4).

For the second part of the claim, let $t \in [0,1]$. Choose $j \in \{1, 2, \ldots, m\}$ such that $t \in [t_{j-1}, t_j]$. Then, using $t - t_{j-1} \leq \frac{1}{m}$ and (4) at the second step,

$$|g(t) - g(t_{j-1})| = (t - t_{j-1}) \frac{|\lambda_j - \lambda_{j-1}|}{t_j - t_{j-1}} < \alpha + \frac{\varepsilon}{9}.$$

Therefore, using this and $t - t_{j-1} \leq \frac{1}{m} < \delta$ at the second step,

$$\begin{aligned} |g(t) - f(t)| &\leq |g(t) - g(t_{j-1})| + |g(t_{j-1}) - f(t_{j-1})| + |f(t_{j-1})(t) - f(t)| \\ &< \left(\alpha + \frac{\varepsilon}{9}\right) + |\lambda_{j-1} - f(t_{j-1})| + \frac{\varepsilon}{9} \leq 2\alpha + \frac{2\varepsilon}{9} < \frac{8\varepsilon}{9}. \end{aligned}$$

The second part of the claim follows.

By (4) and the definition of α, for $j = 1, 2, \ldots, m$ we have

$$\frac{|\lambda_j - \lambda_{j-1}|}{t_j - t_{j-1}} > n + \frac{\varepsilon}{9}.$$

Therefore Lemma 6 provides $\delta_0 > 0$ such that $B_{\delta_0}(g) \cap X_n = \varnothing$. Set $\delta = \min\left(\delta_0, \frac{\varepsilon}{9}\right)$. Then $V = B_\delta(g)$ is a nonempty open set contained in U and with $V \cap X_n = \varnothing$. $\square$

Proof of (2). Part (1) shows that, for every $n \in \mathbb{Z}_{>0}$, the set $C([0,1]) \setminus \overline{X_n}$ is dense in $C([0,1])$. Since $C([0,1])$ is complete, the set

$$G = C([0,1]) \setminus \bigcup_{n=1}^{\infty} \overline{X_n}$$

is a dense G_δ-set. We complete the proof by showing that all functions in $C([0,1]) \setminus \bigcup_{n=1}^{\infty} X_n$ are nowhere differentiable.

Let $f \in C([0,1])$, let $t \in [0,1]$, and assume that $f'(t)$ exists. Then there is $\delta > 0$ such that whenever $s \in [0,1]$ satisfies $0 < |t - s| < \delta$, we have

$$\left| \frac{f(s) - f(t)}{s - t} - f'(t) \right| < 1.$$

This implies that, whenever $|t - s| < \delta$,

$$|f(s) - f(t)| \leq (|f'(t)| + 1)|s - t|.$$

The function

$$s \mapsto \left| \frac{f(s) - f(t)}{s - t} \right|$$

is continuous on the compact set $[0,1] \setminus (t - \delta, t + \delta)$. So there is $M \in [0, \infty)$ such that for all $s \in [0,1]$ with $|t - s| \geq \delta$, we have

$$|f(s) - f(t)| \leq M|s - t|.$$

Choose $n \in \mathbb{Z}_{>0}$ such that $n \geq \max(|f'(t)| + 1, M)$. Then $f \in X_n$. This shows that all functions in $C([0,1]) \setminus \bigcup_{n=1}^{\infty} X_n$ are nowhere differentiable. $\square$