# MATH 617 (WINTER 2024, PHILLIPS): HOMEWORK 2

This homework assignment is due Wednesday 24 January 2024.

**Problem 1** (Problem B; worth two ordinary problems)**.** This is a collection of standard facts about bounded linear operators which are not in Chapter 5 of Rudin, but are in functional analysis books. Please try to do them from scratch, without looking up the proofs in textbooks.

Recall that if $E$ and $F$ are normed vector spaces, then $L(E, F)$ is the set of all bounded (equivalently, continuous) linear maps from $E$ to $F$. (This space is also often called $B(E, F)$.) Further, recall that if $a \in L(E, F)$, then

$$\|a\| = \sup\bigl(\bigl\{\|a\xi\| \colon \xi \in E \text{ and } \|\xi\| \leq 1\bigr\}\bigr).$$

(1) Let $E$ and $F$ be normed vector spaces. Prove that $\|\cdot\|$ is a norm on $L(E, F)$.
(2) Let $E$ be a normed vector space and let $F$ be a Banach space. Prove that $L(E, F)$ is a Banach space.
(3) Let $E_1$, $E_2$, and $E_3$ be normed vector spaces. Let $b \in L(E_1, E_2)$ and let $a \in L(E_2, E_3)$. Prove that $\|ab\| \leq \|a\|\|b\|$.
(4) Let $E$ and $F$ be normed vector spaces, with $E$ finite dimensional. Prove that every linear map $a \colon E \to F$ is bounded. (This is not as straightforward as one might initially think.)

**Problem 2** (Problem 11 in Chapter 5 of Rudin; worth two ordinary problems)**.** Let $\alpha \in (0, 1]$ and let $[a, b] \subset \mathbb{R}$ be a compact interval.

For $f \colon [a, b] \to \mathbb{C}$ define

$$M_{\alpha,f} = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^{\alpha}},$$

and

$$\|f\|_{\alpha} = |f(a)| + M_{\alpha,f} \qquad \text{and} \qquad \|f\|'_{\alpha} = \|f\|_{\infty} + M_{\alpha,f}.$$

Then define

$$\mathrm{Lip}^{\alpha}([a, b]) = \bigl\{f \colon [a, b] \to \mathbb{C} \colon M_{\alpha,f} < \infty\bigr\}.$$

We write $\mathrm{Lip}^{\alpha}$ for short when $[a, b]$ is understood.

Prove that $\mathrm{Lip}^{\alpha}$ is vector space, that $\|\cdot\|_{\alpha}$ and $\|\cdot\|'_{\alpha}$ are norms on $\mathrm{Lip}^{\alpha}$, and that $\mathrm{Lip}^{\alpha}$ is a Banach space with respect to each of these norms.

The functions $f \in \mathrm{Lip}^{\alpha}$ are said to satisfy a *Lipschitz condition of order* $\alpha$, and when $\alpha = 1$ to be *Lipschtiz functions*. The definition makes sense for any metric space $X$ in place of $[a, b]$, and the function spaces are Banach spaces (with the same proofs) whenever in addition $X$ is compact.

*Hint.* The proofs for both norms are essentially the same. One can avoid repeating some of the work by showing that there are constants $c_1, c_2 > 0$ (depending on $a$, $b$, and $\alpha$) such that for all $f \in \mathrm{Lip}^{\alpha}$ we have $c_1\|f\|_{\alpha} \leq \|f\|'_{\alpha} \leq c_2\|f\|_{\alpha}$.

*Date*: 17 January 2024.

**Problem 3** (Problem 16 in Chapter 5 of Rudin)**.** Prove the following theorem (the Closed Graph Theorem). Let $E$ and $F$ be Banach spaces, and let $a\colon E \to F$ be a linear map. Suppose that whenever $(\xi_n)_{n\in\mathbb{Z}_{>0}}$ is a sequence in $E$, $\xi \in E$, $\eta \in F$, and

$$\lim_{n\to\infty} \xi_n = \xi \qquad \text{and} \qquad \lim_{n\to\infty} a\xi_n = \eta,$$

then $a\xi = \eta$. Prove that $a$ is continuous.

Moreover, prove that, without the linearity hypothesis, the statement is false, even for $E = F = \mathbb{R}$.

*Hint.* Make $E \oplus F$ into a Banach space with the standard vector space operations and the norm $\|(\xi,\eta)\| = \|\xi\| + \|\eta\|$ for $\xi \in E$ and $\eta \in F$. (You should check that this formula gives a complete norm on $E \oplus F$, but this is easy.) Define $G \subset E \oplus F$ by

$$G = \big\{(\xi, a\xi)\colon \xi \in E\big\}.$$

Prove that $G$ s a Banach space, and that the restriction to $G$ of the first coordinate projection $E \oplus F \to E$ is bijective and continuous.

(There are other choices for the norm on $E \oplus F$ which work equally well, such as $\|(\xi,\eta)\| = \max(\|\xi\|, \|\eta\|)$.)

For the last part, define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1} & x \neq 0 \\ 0 & x = 0. \end{cases}$$