MATH 617 (WINTER 2024, PHILLIPS): PARTIAL SOLUTIONS TO HOMEWORK 2

This homework assignment is due Wednesday 24 January 2024.

Solutions have not yet been properly proofread.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Problem 1 (Problem B; worth two ordinary problems)**.** This is a collection of standard facts about bounded linear operators which are not in Chapter 5 of Rudin, but are in functional analysis books. Please try to do them from scratch, without looking up the proofs in textbooks.

Recall that if E and F are normed vector spaces, then $L(E, F)$ is the set of all bounded (equivalently, continuous) linear maps from E to F. (This space is also often called $B(E, F)$.) Further, recall that if $a \in L(E, F)$, then

$$\|a\| = \sup\big(\big\{\|a\xi\| \colon \xi \in E \text{ and } \|\xi\| \leq 1\big\}\big).$$

(1) Let E and F be normed vector spaces. Prove that $\|\cdot\|$ is a norm on $L(E, F)$.
(2) Let E be a normed vector space and let F be a Banach space. Prove that $L(E, F)$ is a Banach space.
(3) Let E_1, E_2, and E_3 be normed vector spaces. Let ~~$a \in L(E_1, E_2)$ and let $b \in L(E_2, E_3)$~~ $b \in L(E_1, E_2)$ and let $a \in L(E_2, E_3)$. Prove that $\|ab\| \leq \|a\|\|b\|$.
(4) Let E and F be normed vector spaces, with E finite dimensional. Prove that every linear map $a \colon E \to F$ is bounded. (This is not as straightforward as one might initially think.)

Proof of Part (1). ~~No proof has been written yet, but this is just calculations.~~ We mostly use the characterization of $\|a\|$ (done in class, but stated here in two parts for convenience of writing)

$$\|a\| = \inf\big(\big\{M \in [0, \infty) \colon \text{for all } \xi \in E \text{ we have } \|a\xi\| \leq M\|\xi\|\big\}\big) \tag{1}$$

and, for all $\xi \in E$,

$$\|a\xi\| \leq \|a\|\|\xi\|. \tag{2}$$

Nonemptiness of the set in (1) is the definition of boundedness for a linear map, and (2) says this set contains its infimum. (It is necessary to specify $M \geq 0$ in (1), since otherwise if $E = \{0\}$ and $a \in L(E, F)$ is the zero operator, then $\|a\| = -\infty$.)

It is obvious that if $a \in L(E, F)$ then $\|a\| \geq 0$.

Suppose $\|a\| = 0$. By (2), for any $\xi \in E$ we have $\|a\xi\| \leq \|a\|\|\xi\| = 0$, so $a\xi = 0$. Since this holds for all $\xi \in E$, we get $a = 0$.

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For subadditivity, let $a, b \in L(E, F)$. For all $\xi \in E$ we have, using subadditivity of the norm on F at the second step and (2) at the third step,

$$\|(a+b)\xi\| = \|a\xi + b\xi\| \leq \|a\xi\| + \|b\xi\| \leq \|a\|\|\xi\| + \|b\|\|\xi\| = \big(\|a\| + \|b\|\big)\|\xi\|.$$

Since this is true for all $\xi \in E$, by (1) we get $\|a+b\| \leq \|a\| + \|b\|$.

To prove homogeneity, for $\lambda \in \mathbb{C}$ and $a \in L(E, F)$ we calculate:

$$\|\lambda a\| = \sup_{\|\xi\| \leq 1\|} \|(\lambda a)\xi\| = \sup_{\|\xi\| \leq 1\|} \|\lambda \cdot (a\xi)\| = \sup_{\|\xi\| \leq 1\|} |\lambda| \|a\xi\| = |\lambda| \sup_{\|\xi\| \leq 1\|} \|a\xi\| = |\lambda|\|a\|.$$

This completes the solution. □

Alternate proof of homogeneity. For all $\xi \in E$ we have, using homogeneity of the norm on F at the second step and (2) at the third step,

$$\|(\lambda a)\xi\| = \|\lambda \cdot (a\xi)\| = |\lambda|\|a\xi\| \leq |\lambda|\|a\|\|\xi\|.$$

Therefore, by (1),

$$\|\lambda a\| \leq |\lambda|\|a\|. \tag{3}$$

If $\lambda = 0$, nonnegativity of the norm on $L(E, F)$ gives $\|\lambda a\| = 0 = |\lambda|\|a\|$. Otherwise, apply (3) with λ^{-1} in place of λ and λa in place of a, getting $\|a\| \leq |\lambda|^{-1}\|\lambda a\|$. Combining this with (3) gives $\|\lambda a\| = |\lambda|\|a\|$. □

Proof of (2). Let $(a_n)_{n \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in $L(E, F)$. Let $\xi \in E$. For $m, n \in \mathbb{Z}_{>0}$, we have

$$\|a_m \xi - a_n \xi\| = \|(a_m - a_n)\xi\| \leq \|\xi\|\|a_m - a_n\|,$$

It is now immediate that $(a_n \xi)_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in F. Since F is complete, $\lim_{n \to \infty} a_n \xi$ exists. Denote this limit by $a\xi$.

Using continuity of addition and scalar multiplication, and linearity of a_n for $n \in \mathbb{Z}_{>0}$, one checks immediately that a is linear.

In any normed vector space, we have $\big|\|\xi\| - \|\eta\|\big| \leq \|\xi - \eta\|$. Applying this to $L(E, F)$, it is immediate that $(\|a_n\|)_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathbb{R}$. Therefore $M = \sup_{n \in \mathbb{Z}_{>0}} \|a_n\|$ is finite. For $\xi \in E$, we have

$$\|a\xi\| = \lim_{n \to \infty} \|a_n \xi\| \leq \limsup_{n \to I \; n \to \infty} \|a_n\|\|\xi\| \leq M\|\xi\|.$$

Thus a is bounded.

It remains to show that $\lim_{n \to \infty} \|a_n - a\| = 0$. (Without doing this, the proof is not complete.) So let $\varepsilon > 0$. Choose $N \in \mathbb{Z}_{>0}$ such that for all $m, n \geq N$ we have $\|a_m - a_n\| < \frac{\varepsilon}{2}$. Let $n \in \mathbb{Z}_{>0}$ satisfy $n \geq N$. Then for any $\xi \in E$ we have

$$\|(a_n - a)\xi\| = \|a_n \xi - a\xi\| = \lim_{m \to \infty} \|a_n \xi - a_m \xi\| \leq \limsup_{n \to I} \|a_n - a_m\|\|\xi\| \leq \frac{\varepsilon}{2}\|\xi\|.$$

This shows that $\|a_n - a\| \leq \frac{\varepsilon}{2} < \varepsilon$. Thus $\lim_{n \to \infty} \|a_n - a\| = 0$. □

Proof of (3). No proof has been written yet, but this is just a calculation. □

Proof of (4). Let $\eta_1, \eta_2, \ldots, \eta_n$ form a basis for E.

Define

$$S = \big\{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n \colon \|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_2 = 1\big\},$$

which is a compact subset of $\mathbb{C}^n$, and define a continuous function $f\colon S \to [0, \infty)$ by

$$f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\| \sum_{j=1}^{n} \alpha_j \eta_j \right\|.$$

Set ~~$c = \inf(\{f(\alpha)\colon \alpha \in S)$~~ $c = \inf(\{f(\alpha)\colon \alpha \in S\})$. Since $\eta_1, \eta_2, \ldots, \eta_n$ are linearly independent, $0 \notin \mathrm{Ran}(f)$, so $c > 0$. Set

$$M = \frac{1}{c} \sum_{j=1}^{n} \|a\eta_j\|.$$

We will show $\|a\| \leq M$.

Let $\xi \in E$ satisfy $\|\xi\| = 1$. Then there are $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{C}$ such that $\xi = \sum_{j=1}^{n} \beta_j \eta_j$. Set $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and for $j = 1, 2, \ldots, n$ set $\alpha_j = \|\beta\|_2^{-1} \beta_j$. Then $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$. Therefore $\big\| \|\beta\|_2^{-1} \xi \big\| \geq c$. Since $\|\xi\| = 1$, we get

$$\frac{1}{c} \geq \|\beta\|_2 \geq \max{}_{1leqj\leq n \, 1 \leq j \leq n} |\beta_j|,$$

whence

$$\|a\xi\| = \left\| \sum_{j=1}^{n} \beta_j a \eta_j \right\| \leq \left(\max_{1leqj\leq n} |\beta_j| \right) \sum_{j=1}^{n} \|a_j \eta_j\| \leq \frac{1}{c} \sum_{j=1}^{n} \|a\eta_j\| = M.$$

For general nonzero $\xi \in E$, applying this to $\|\xi\|^{-1}\xi$ gives $\|a\xi\| \leq M\|\xi\|$. This is also true for $\xi = 0$. □

Remark 1. It is common to write, "let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be a basis for E", but this means something different. For example, $\big\{(1, 0),\, (1, 0),\, (0, 1)\big\}$ is a basis for $\mathbb{C}^2$, because

$$\big\{(1, 0),\, (1, 0),\, (0, 1)\big\} = \big\{(1, 0),\, (0, 1)\big\}.$$

Problem 2 (Problem 11 in Chapter 5 of Rudin; worth two ordinary problems)**.** Let ~~$0 < \alpha \leq 1$~~ $\alpha \in (0, 1]$ and let $[a, b] \subset \mathbb{R}$ be a compact interval.

For $f\colon [a, b] \to \mathbb{C}$ define

$$M_{\alpha,f} = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^{\alpha}},$$

and

$$\|f\|_{\alpha} = |f(a)| + M_{\alpha,f} \qquad \text{and} \qquad \|f\|'_{\alpha} = \|f\|_{\infty} + M_{\alpha,f}.$$

Then define

$$\mathrm{Lip}^{\alpha}([a, b]) = \big\{ f\colon [a, b] \to \mathbb{C}\colon M_{\alpha,f} < \infty \big\}.$$

We write Lip^{α} for short when $[a, b]$ is understood.

Prove that Lip^{α} is vector space, that $\|\cdot\|_{\alpha}$ and $\|\cdot\|'_{\alpha}$ are norms on Lip^{α}, and that Lip^{α} is a Banach space with respect to each of these norms.

The functions $f \in \mathrm{Lip}^{\alpha}$ are said to satisfy a *Lipschitz condition of order* α, and when $\alpha = 1$ to be *Lipschtiz functions*. The definition makes sense for any metric space X in place of $[a, b]$, and the function spaces are Banach spaces (with the same proofs) whenever in addition X is compact.

Hint. The proofs for both norms are essentially the same. One can avoid repeating some of the work by showing that there are constants $c_1, c_2 > 0$ (depending on a, b, and α) such that for all $f \in \mathrm{Lip}^\alpha$ we have ~~$c_1\|f\|_\alpha \leq \|f\|'_\alpha \leq c_2\|f\|_\alpha$~~$c_1\|f\|_\alpha \leq \|f\|'_\alpha \leq c_2\|f\|_\alpha$.

The solution is presented as a number of lemmas, organized so as to minimize the work.

Lemma 2. Let $f\colon [a,b] \to \mathbb{C}$ and suppose that $M_{\alpha,f} < \infty$. Then f is continuous.

The proof is standard in earlier courses, and is omitted.

Lemma 3. Let $f \in \mathrm{Lip}^\alpha$. Then $\|f\|_\alpha \leq \|f\|'_\alpha$.

Proof. This is immediate. □

Lemma 4. Let $c = 1 + \max(1,\, (b-a)^\alpha)$. Then $\|f\|'_\alpha \leq c\|f\|_\alpha$ for all $f \in \mathrm{Lip}^\alpha$.

Proof. Let $f \in \mathrm{Lip}^\alpha$. For any $t \in [a,b]$, we have $|t-a| \leq b-a$, so

$$\begin{aligned} |f(t)| &\leq |f(a)| + |f(t) - f(a)| \leq |f(a)| + M_{\alpha,f}|t-a|^\alpha \\ &\leq |f(a)| + M_{\alpha,f}(b-a)^\alpha \leq \max(1,\, (b-a)^\alpha)\|f\|_\alpha. \end{aligned}$$

Therefore $\|f\|_\infty \leq \max(1,\, (b-a)^\alpha)\|f\|_\alpha$. So

$$\|f\|'_\alpha = \|f\|_\infty + M_{\alpha,f} \leq \max(1,\, (b-a)^\alpha)\|f\|_\alpha + \|f\|_\alpha = c\|f\|_\alpha.$$

This completes the proof. □

Lemma 5. The set Lip^α is a vector space, and $\|\cdot\|_\alpha$ and $\|\cdot\|'_\alpha$ are norms on Lip^α.

Proof. If $f \in \mathrm{Lip}^\alpha$ then $M_{\alpha,f} < \infty$ by definition, and then clearly $\|f\|_\infty < \infty$~~by Lemma 2, so $\|f\|'_\alpha < \infty$. Therefore $\|f\|_\alpha < \infty$~~. Therefore $\|f\|'_\alpha < \infty$ by Lemma ~~3~~4.

~~A calculation (which we omit) shows that for any functions $f, g\colon [a,b] \to \mathbb{C}$~~ We claim that if $f, g\colon [a,b] \to \mathbb{C}$ and $\lambda \in \mathbb{C}$~~we have~~, then

$$M_{\alpha,f+g} \leq M_{\alpha,f} + M_{\alpha,g} \qquad \text{and} \qquad M_{\alpha,\lambda f} = |\lambda| \cdot M_{\alpha,f}.$$

For the first,

$$\begin{aligned} M_{\alpha,f+g} &= \sup_{s \neq t} \frac{|f(s) + g(s) - f(t) - g(t)|}{|s-t|^\alpha} \leq \sup_{s\neq t}\left(\frac{|f(s)-f(t)|}{|s-t|^\alpha} + \frac{|g(s)-g(t)|}{|s-t|^\alpha}\right) \\ &\leq \sup_{s\neq t} \frac{|f(s)-f(t)|}{|s-t|^\alpha} + \sup_{s\neq t}\frac{|g(s)-g(t)|}{|s-t|^\alpha} = M_{\alpha,f} + M_{\alpha,g}. \end{aligned}$$

For the second,

$$\begin{aligned} M_{\alpha,\lambda f} &= \sup_{s\neq t} \frac{|\lambda f(s) - \lambda f(t)|}{|s-t|^\alpha} = \sup_{s \neq t} |\lambda| \left(\frac{|f(s)-f(t)|}{|s-t|^\alpha}\right) \\ &= |\lambda| \sup_{s\neq t}\left(\frac{|f(s)-f(t)|}{|s-t|^\alpha}\right) = |\lambda| \cdot M_{\alpha,f}. \end{aligned}$$

The claim is proved.

The known properties of $|\cdot|$ and $\|\cdot\|_\infty$ now immediately imply

$$\|f+g\|_\alpha \leq \|f\|_\alpha + \|g\|_\alpha \qquad \text{and} \qquad \|\lambda f\|_\alpha = |\lambda| \cdot \|f\|_\alpha,$$

and

$$\|f+g\|'_\alpha \le \|f\|'_\alpha + \|g\|'_\alpha \qquad \text{and} \qquad \|\lambda f\|'_\alpha = |\lambda| \cdot \|f\|'_\alpha.$$

It immediately follows that Lip^α is closed under addition and scalar multiplication, so is a vector space. To show that $\|\cdot\|_\alpha$ and $\|\cdot\|'_\alpha$ are norms, it only remains to show that if $\|f\|_\alpha = 0$ or $\|f\|'_\alpha = 0$ then $f = 0$. By Lemma 3 we need only consider the case $\|f\|_\alpha = 0$. If $\|f\|_\alpha = 0$ then clearly $M_{\alpha,f} = 0$, from which it follows that $|f(s) - f(t)| = 0$ whenever $s \neq t$. That is, f is constant. But $\|f\|_\alpha = 0$ also implies $|f(a)| = 0$, whence $f = 0$. □

Lemma 6. The space Lip^α is complete in $\|\cdot\|'_\alpha$.

Proof. Let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in Lip^α with respect to $\|\cdot\|'_\alpha$. Since $\|f\|_\infty \le \|f\|'_\alpha$ for any f, it follows from Lemma 2 that $(f_n)_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $C([a,b])$. Therefore there is $f \in C([a,b])$ such that $\|f_n - f\|_\infty \to 0$, that is, $(f_n)_{n\in\mathbb{Z}_{>0}}$ converges uniformly to f.

Set $M = \sup_{n\in\mathbb{Z}_{>0}} \|f_n\|'_\alpha$. Since $(f_n)_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence with respect to $\|\cdot\|'_\alpha$, we get $M < \infty$. Now let $s, t \in [a,b]$ with $s \neq t$. Since $f_n(s) \to f(s)$ and $f_n(t) \to f(t)$, we get

$$\frac{|f(s)-f(t)|}{|s-t|^\alpha} = \lim_{n\to\infty} \frac{|f_n(s)-f_n(t)|}{|s-t|^\alpha} \le M.$$

Since s and t are arbitrary, we get $M_{\alpha,f} \le M < \infty$. Therefore $f \in \mathrm{Lip}^\alpha$.

We must still show that $\lim_{n\to\infty} \|f_n - f\|'_\alpha = 0$. Let $\varepsilon > 0$. Choose N so large that whenever $m, n \ge N$ then $\|f_m - f_n\|'_\alpha < \frac{1}{3}\varepsilon$. We are going to show that $n \ge N$ implies $\|f_n - f\|'_\alpha < \varepsilon$, and the main part is to estimate $M_{\alpha,\, f_n - f}$. So let $n \ge N$. Let $s, t \in [a,b]$ with $s \neq t$. Then

$$\frac{|[f_n(s)-f(s)]-[f_n(t)-f(t)]|}{|s-t|^\alpha} = \lim_{m\to\infty} \frac{|[f_n(s)-f_m(s)]-[f_n(t)-f_m(t)]|}{|s-t|^\alpha}$$

$$\le \sup_{m\ge N} M_{\alpha,\, f_n-f_m} \le \tfrac{1}{3}\varepsilon.$$

$$\frac{|[f_n(s)-f(s)]-[f_n(t)-f(t)]|}{|s-t|^\alpha} = \lim_{m\to\infty} \frac{|[f_n(s)-f_m(s)]-[f_n(t)-f_m(t)]|}{|s-t|^\alpha}$$

$$\le \sup_{m\ge N} M_{\alpha,\, f_n-f_m} \le \tfrac{1}{3}\varepsilon.$$

Taking the supremum over all s and t such that $s \neq t$, we get $M_{\alpha,\, f_n-f} \le \frac{1}{3}\varepsilon < \frac{2}{3}\varepsilon$. Therefore

$$\|f_n - f\|'_\alpha = \|f_n - f\|_\infty + M_{\alpha,\, f_n-f} < \tfrac{1}{3}\varepsilon + \tfrac{2}{3}\varepsilon = \varepsilon.$$

That is, $n \ge N$ implies $\|f_n - f\|'_\alpha < \varepsilon$. □

Remark 7. Proving Lemma 6 without using completeness of $C([a,b])$ requires only one or two more sentences, since continuity of the limit function is not used. This change eliminates the need for Lemma 2.

Remark 8. It is almost as easy to prove directly that Lip^α is complete in $\|\cdot\|_\alpha$. One extra piece of reasoning is required, which is in any case contained in the proof of Lemma 4 below.

Remark 9. In the proof of Lemma 6, it is essential to prove that $\lim_{n\to\infty} \|f_n - f\|'_\alpha = 0$. Without this, the lemma has not been proved.

Let $c = 1 + \max(1, (b-a)^\alpha)$. Then $\|f\|'_\alpha \leq c\|f\|_\alpha$ for all $f \in \mathrm{Lip}^\alpha$. Let $f \in \mathrm{Lip}^\alpha$. For any $t \in [a, b]$, we have $|t - a| \leq b - a$, so

$$|f(t)| \leq |f(a)| + |f(t) - f(a)| \leq |f(a)| + M_{\alpha,f}|t-a|^\alpha$$

$$\leq |f(a)| + M_{\alpha,f}(b-a)^\alpha \leq \max(1, (b-a)^\alpha)\|f\|_\alpha.$$

Therefore $\|f\|_\infty \leq \max(1, (b-a)^\alpha)\|f\|_\alpha$. So

$$\|f\|'_\alpha = \|f\|_\infty + M_{\alpha,f} \leq \max(1, (b-a)^\alpha)\|f\|_\alpha + \|f\|_\alpha = c\|f\|_\alpha.$$

This completes the proof.

Proposition 10. The space Lip^α is a Banach space in the norm $\|\cdot\|_\alpha$ and also in the norm $\|\cdot\|'_\alpha$.

Proof. For $\|\cdot\|'_\alpha$, combine Lemma 5 and Lemma 6. For $\|\cdot\|_\alpha$, according to Lemma 5 we need only verify that Lip^α is complete in this norm. So let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in Lip^α with respect to $\|\cdot\|_\alpha$. It follows from Lemma 4 that $(f_n)_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence with respect to $\|\cdot\|'_\alpha$. Lemma 6 provides $f \in \mathrm{Lip}^\alpha$ such that $\lim_{n\to\infty} \|f - f_n\|'_\alpha = 0$. It is now immediate from Lemma 3 that $\lim_{n\to\infty} \|f - f_n\|_\alpha = 0$. □

Remark 11. Let E be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E are called *equivalent* if there are $c_1, c_2 > 0$ such that $c_1\|\xi\|_1 \leq \|\xi\|_2 \leq c_2\|\xi\|_1$ for all $\xi \in E$. It is easily checked that this relation is an equivalence relation. Also, one checks easily that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if both the identity maps

$$\mathrm{id}_E \colon (E, \|\cdot\|_1) \to (E, \|\cdot\|_2) \qquad \text{and} \qquad \mathrm{id}_E \colon (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$$

are continuous.

Lemma 3 and Lemma 4 show that $\|\cdot\|_\alpha$ and $\|\cdot\|'_\alpha$ are equivalent, and the proof of Proposition 10 shows that a vector space which is complete in some norm is also complete in any equivalent norm.

Problem 3 (Problem 16 in Chapter 5 of Rudin)**.** Prove the following theorem (the Closed Graph Theorem). Let E and F be Banach spaces, and let $a \colon E \to F$ be a linear map. Suppose that whenever $(\xi_n)_{n\in\mathbb{Z}_{>0}}$ is a sequence in E, $\xi \in E$, $\eta \in F$, and

$$\lim_{n\to\infty} \xi_n = \xi \qquad \text{and} \qquad \lim_{n\to\infty} a\xi_n = \eta,$$

then $a\xi = \eta$. Prove that a is continuous.

Moreover, prove that, without the linearity hypothesis, the statement is false, even for $X = Y = \mathbb{R}$ $E = F = \mathbb{R}$.

Hint. Make $E \oplus F$ into a Banach space with the standard vector space operations and the norm $\|(\xi, \eta)\| = \|\xi\| + \|\eta\|$ for $\xi \in E$ and $\eta \in F$. (You should check that this formula gives a complete norm on $E \oplus F$, but this is easy.) Define $G \subset E \oplus F$ by

$$G = \{(\xi, a\xi) \colon \xi \in E\}.$$

Prove that G s a Banach space, and that the restriction to G of the first coordinate projection $E \oplus F \to E$ is bijective and continuous.

(There are other choices for the norm on $E \oplus F$ which work equally well, such as $\|(\xi, \eta)\| = \max(\|\xi\|, \|\eta\|)$.)

For the last part, define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Solution to the first part. We use the notation of the hint.

It is well known that $E \oplus F$ as described is a vector space. It is easy to check that the formula $\|(\xi, \eta)\| = \|\xi\| + \|\eta\|$ does in fact define a norm on $E \oplus F$; the proof is omitted. Moreover, one easily checks that, using this norm, one has $\lim_{n\to\infty}(\xi_n, \eta_n) = (\xi, \eta)$ if and only if $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} \eta_n = \eta$.

Define $p\colon E \oplus F \to E$ and $q\colon E \oplus F \to F$ by $p(\xi, \eta) = \xi$ and $q(\xi, \eta) = \eta$ for all $\xi \in E$ and $\eta \in F$. These are well known to be linear. Moreover, clearly $\|p(\xi, \eta)\| \leq \|(\xi, \eta)\|$ and $\|q(\xi, \eta)\| \leq \|(\xi, \eta)\|$. In particular, p and q are continuous.

We prove that $E \oplus F$ is complete. Consider a Cauchy sequence in $E \oplus F$. It has the form $((\xi_n, \eta_n))_{n \in \mathbb{Z}_{>0}}$ with ~~$\xi_n \in \varnothing$~~ $\xi_n \in E$ and $\eta_n \in F$ for all $n \in \mathbb{Z}_{>0}$. Since p and q reduce norms, it is immediate that $(\xi_n)_{n\in\mathbb{Z}_{>0}}$ and $(\eta_n)_{n\in\mathbb{Z}_{>0}}$ are Cauchy sequences. Therefore $\xi = \lim_{n\to\infty} \xi_n$ and $\eta = \lim_{n\to\infty} \eta_n$ exist. To complete the proof of completeness, we show that $\lim_{n\to\infty}(\xi_n, \eta_n) = (\xi, \eta)$. But this is immediate from the observation at the end of the second paragraph.

Let G be as in the hint. We are assuming that for every sequence $(\xi_n)_{n\in\mathbb{Z}_{>0}}$ in E such that both $\xi = \lim_{n\to\infty} \xi_n$ and $\eta = \lim_{n\to\infty} a\xi_n$ exist, one has $a\xi = \eta$. By the observation at the end of the second paragraph, this says that if $(\gamma_n)_{n\in\mathbb{Z}_{>0}}$ is a sequence in G such that $\lim_{n\to\infty} \gamma_n$ exists, then $\lim_{n\to\infty} \gamma_n \in G$. Thus G is closed. It is immediate from the linearity of a that G is a vector subspace of $E \oplus F$. Thus G is a Banach space.

The map $p|_G\colon G \to E$ is clearly bijective. Since it is continuous and linear, the Open Mapping Theorem implies that $(p|_G)^{-1}$ is continuous. Therefore $a = (q|_G) \circ (p|_G)^{-1}$ is continuous. □

It is a standard fact, which you need not reprove in the future, that $E \oplus F$ as defined here is a Banach space and that the projections p and q are linear and norm reducing.

Proof for the nonlinear example. It is obvious that f is not continuous at 0. To see that the graph $G \subset \mathbb{R}^2$ of f is closed, define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = xy$, and write $G = \{(0, 0)\} \cup F^{-1}(\{1\})$. This is a union of two closed sets because F is continuous. □

Since many people have used sequences in the past, here is a proof in terms of sequences.

Second proof of the nonlinear example. It is obvious that f is not continuous at 0. To see that the graph $G \subset \mathbb{R}^2$ of f is closed, let $((x_n, y_n))_{n\in\mathbb{Z}_{>0}}$ be a sequence in G, and suppose that $\lim_{n\to\infty}(x_n, y_n) = (x, y)$. We need to prove that $(x, y) \in G$.

We have, in particular, $\lim_{n\to\infty} x_n = x$. First suppose that $x \neq 0$. Then there is $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ we have $x_n \neq 0$. For such values of n, we have $y_n = \frac{1}{x_n}$. Since the function ~~$h(x) = \frac{1}{x}$~~ $h(t) = \frac{1}{t}$ is continuous at ~~x_0~~ x, it follows that $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$. Therefore $y = \frac{1}{x}$, whence $(x, y) \in G$.

Now suppose that $x = 0$. If $x_n = 0$ for all but finitely many n, then $y_n = 0$ for all but finitely many n, so $\lim_{n\to\infty}(x_n, y_n) = (0, 0)$, which is in G. Otherwise, we show that $\lim_{n\to\infty} y_n$ does not exist, which is a contradiction. Let $M \in (0, \infty)$; we show that for all $N \in \mathbb{Z}_{>0}$ there is $n \in \mathbb{Z}_{>0}$ with $n \geq N$ and such that $|y_n| > M$. Apply the definition of $\lim_{n\to\infty} x_n = 0$ with $\frac{1}{M}$ in place ε, getting $N_0 \in \mathbb{Z}_{>0}$. Since there are infinitely many $n \in \mathbb{Z}_{>0}$ such that $x_n \neq 0$, we can find $n \in \mathbb{Z}_{>0}$ such that $x_n \neq 0$ and $n > \max(N, N_0)$. Then $n \geq N$ and $|y_n| > \frac{1}{\varepsilon} = M$, as desired. This shows that $\lim_{n\to\infty} y_n$ does not exist. □

Remark 12. The following claim has been made in the past: If $(x, y) \in \mathbb{R}^2 \setminus G$ and

$$\delta = \min(\|(x, y)\|, \|(x, y) - (x, 1/x)\|, \|(x, y) - (1/y, y)\|)$$

(the minimum of the distances from (x, y) to the three points $(0, 0)$, $(x, 1/x)$, and $(1/y, y)$), then the open ball $B_\delta(x, y)$ does not intersect G. This is not true. Take $x = y = \frac{5}{8}$. Then $(1, 1) \in G$ and

$$\delta = \min\left(\frac{5\sqrt{2}}{8}, \frac{39}{40}\right) > \frac{3\sqrt{2}}{8} = \|(x, y) - (1, 1)\|,$$

so $(1, 1) \in B_\delta(x, y) \cap G$.