MATH 617 (WINTER 2024, PHILLIPS): HOMEWORK 3

This homework assignment is due Wednesday 31 January 2024.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Problem 1 (Problem 6 in Chapter 5 of Rudin)**.** Let H be a Hilbert space, let $M \subset H$ be a closed subspace, and let $\omega_0 \in M^*$. Prove that there is a *unique* $\omega \in H^*$ such that $\omega|_M = \omega_0$ and $\|\omega\| = \|\omega_0\|$. Moreover, prove that ω vanishes on $M^{\perp}$.

Problem 2 (Problem 18 in Chapter 5 of Rudin)**.** Let E be a normed vector space, let F be a Banach space, and let $(a_n)_{n \in \mathbb{Z}_{>0}}$ be a bounded sequence in $L(E, F)$. Suppose that there is a dense set $S \subset E$ such that $\lim_{n \to \infty} a_n \xi$ exists for all $\xi \in S$. Prove that $\lim_{n \to \infty} a_n \xi$ exists for all $\xi \in E$.

Problem 3 (An expansion of Problem 17 in Chapter 5 of Rudin)**.** (This problem is worth two ordinary problems.) Let μ be a n onzero positive measure on a measurable space X. Let $p \in [1, \infty)$. For $f \in L^{\infty}(X, \mu)$, let $m(f) \colon L^p(X, \mu) \to L^p(X, \mu)$ be defined by $m(f)(\xi)(x) = f(x)\xi(x)$, that is, $m(f)$ is the multiplication operator by f.

(1) Prove that $\|m(f)\| \leq \|f\|_{\infty}$ for all $f \in L^{\infty}(X, \mu)$.
(2) Prove that $\|m(f)\| = \|f\|_{\infty}$ for all $f \in L^{\infty}(X, \mu)$ if and only if μ is semifinite.
(3) Assume that μ is semifinite. Give, with proof, a characterization in terms of f of those $f \in L^{\infty}(X, \mu)$ for which the operator $m(f)$ is surjective.
(4) Assume that μ is semifinite. Let $f \in L^{\infty}(X, \mu)$, and suppose that $m(f)$ is surjective. Prove that $m(f)$ is injective.
(5) Give, with proof, an example of a finite measure μ on a measurable space X and $f \in L^{\infty}(X, \mu)$ such that that $m(f)$ is injective but not surjective.

Recall that a measure μ on X is called *semifinite* if for every measurable set $E \subset X$ with $\mu(E) > 0$, there is a measurable set $F \subset E$ with $0 < \mu(F) < \infty$.

Example 1. Here are some examples of measures which are semifinite and some which are not. (This isn't an exercise.)

(1) Every σ-finite measure is semifinite.
(2) Counting measure on $\mathbb{R}$ is semifinite but not σ-finite.
(3) On any set X take the measurable sets to be $\varnothing$ and X, and take $\mu(\varnothing) = 0$ and $\mu(X) = \infty$. Then μ is not semifinite.
(4) On $\mathbb{R}$ take the measurable sets to be the countable sets and their complements. Take $\mu(E) = 0$ if E is countable and $\mu(E) = \infty$ if $\mathbb{R} \setminus E$ is countable. Then μ is not semifinite.

I know of no real use for measures which are not semifinite.

Problem 4 (Problem 8(c) in Chapter 5 of Rudin)**.** Let E be a normed vector space, and let $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in E. Suppose that $\lim_{n \to \infty} \omega(\xi_n)$ exists for all $\omega \in E^*$. Prove that $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ is bounded.

Date: 24 January 2024.