MATH 617 (WINTER 2024, PHILLIPS): HOMEWORK 4

This homework assignment is due Wednesday 7 February 2024.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solution file.

Problem 1 (Problem 1 in Chapter 5 of Rudin)**.** Let $X = \{a, b\}$, let $\alpha, \beta \in (0, \infty)$, and let $\mu_{\alpha,\beta}$ be the measure on X such that that $\mu_{\alpha,\beta}(\{a\}) = \alpha$ and $\mu_{\alpha,\beta}(\{b\}) = \beta$. In this problem, we use the spaces $L^p(X, \mu, \mathbb{R})$ of *real* valued L^p functions on X modulo functions vanishing almost everywhere. (In this problem, the only set of measure zero will be $\varnothing$.)

(1) For $p \in (0, \infty]$ describe the closed unit ball of $L^p(X, \mu_{1,1}, \mathbb{R})$. In particular, show that it is convex if and only if $p \in [1, \infty]$, determine for which values of p it is a circle, and determine for which values of p it is a square. Draw pictures of these unit balls for representative choices of p, such as $p = 1, 2, \infty$ and some value of p in each of the intervals $(0, 1)$, $(1, 2)$, and $(2, \infty)$.

(2) Describe what happens to your solution to part (1) for $\alpha \neq \beta$, say for $\mu_{1,1/2}$ in place of $\mu_{1,1}$.

Problem 2 (Problems 2 and 3 in Chapter 5 of Rudin)**.** This problem counts as two regular problems.

Three problems on convexity:

(1) Let E be a Banach space. Prove that the closed unit ball B of E is convex, that is, if $\xi, \eta \in B$ and $\alpha \in [0, 1]$ then $(1 - \alpha)\xi + \alpha\eta \in B$.

(2) Let (X, μ) be a measure space, and let $p \in (1, \infty)$. Prove that the closed unit ball B of E is strictly convex, that is, if $\xi, \eta \in B$ are distinct and $\alpha \in (0, 1)$ then $\|(1-\alpha)\xi + \alpha\eta\| < 1$. (The statement means that the surface of the closed unit ball contains no straight line segments. You will need the criterion for equality in the triangle inequality for $\| \cdot \|_p$.)

(3) Let E be any nontrivial space of the form $C(X)$, $L^1(X, \mu)$, or $L^\infty(X, \mu)$. Prove that the closed unit ball B of E is not strictly convex. (Part of the problem is to determine what "trivial" means. If X has only one point then E is certainly trivial for the purposes of this problem, but there are other ways for $L^1(X, \mu)$ and $L^\infty(X, \mu)$ to be trivial.)

Problem 3 (Problem C)**.** This problem counts as two regular problems.

Let (X, μ) be a measure space. For a measurable function f on X and $\alpha > 0$, define

$$\lambda_f(\alpha) = \mu\big(\{x \in X \colon |f(x)| > \alpha\}\big).$$

For $p \in [1, \infty)$ define

$$C_p(f) = \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha)\right)^{1/p},$$

and define $L^p_{\mathrm{w}}(\mu)$ ("weak $L^p(\mu)$") to be the set of measurable functions f on X such that $C_p(f) < \infty$ (modulo equality almost everywhere, as usual).

Prove the following:

(1) $C_p(f) \leq \|f\|_p$ and $L^p(\mu) \subset L^p_{\mathrm{w}}(\mu)$.

Date: 31 January 2024.

(2) $L^p_{\mathrm{w}}(\mu)$ is a vector space. (Hint: prove $C_p(f+g) \leq 2\big(C_p(f)^p + C_p(g)^p\big)^{1/p}$ and $C_p(\beta f) = |\beta| C_p(f)$.)
(3) C_1 need not be a norm.
(4) If $\mu(X) < \infty$, then $L^p_{\mathrm{w}}(\mu) \subset L^r(\mu)$ whenever $1 \leq r < p$.

Remark 1. Some remarks on Problem 3:

(1) Most of this problem is taken from Folland's book. (I didn't check the inequality in the hint in part (2) for correctness, but something similar is certainly true.)
(2) The notation is mine, and is probably nonstandard. In another book, for the case of Lebesgue measure on $\mathbb{R}$, $L^p_{\mathrm{w}}(\mu)$ is called $L(p,\infty)$, and there are "L^p type spaces" $L(p,r)$ for $1 \leq r \leq \infty$.
(3) This is from my reading elsewhere (which actually only specifically talked about Lebesgue measure on $\mathbb{R}$). None of the functions C_p is a norm. But if $p > 1$, then there are norms on the spaces $L^p_{\mathrm{w}}(\mu)$ which are equivalent to C_p (in the sense we usually apply to norms). With these norms, the spaces $L^p_{\mathrm{w}}(\mu)$ are Banach spaces. The space $L^1_{\mathrm{w}}(\mu)$ is a topological vector space, but metrizability and completeness were not mentioned.