MATH 617 (WINTER 2024, PHILLIPS): HOMEWORK 6

Problem 1 (Problem 5 in Chapter 7 of Rudin)**.** This problem counts as five regular problems.

Let M be the Banach space of all complex Borel measures on $\mathbb{R}$. Recall that $\|\mu\| = |\mu|(\mathbb{R})$ for $\mu \in M$.

Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ be the function $s(x, y) = x + y$ for $x, y \in \mathbb{R}$. For $\mu, \nu \in M$, define $\mu * \nu$ to be the set function given by $(\mu * \nu)(E) = (\mu \times \nu)(s^{-1}(E))$ for every Borel set $E \subset \mathbb{R}$. (The function $\mu * \nu$ is called the *convoluton* of μ and ν.)

(1) Let $\mu, \nu \in M$. Prove that $\mu * \nu$ is a complex Borel measure on $\mathbb{R}$ which satisfies $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$.

(2) Let $\mu, \nu \in M$. Prove that $\mu * \nu$ is the unique measure $\lambda \in M$ which satisfies

$$\int_{\mathbb{R}} f \, d\lambda = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x + y) \, d\mu(x) \right) d\nu(y)$$

for all $f \in C_0(\mathbb{R})$.

(3) Prove that the operation $(\mu, \nu) \mapsto \mu * \nu$, from $M \times M$ to M, is commutative, associative, distributes over addition, and satisfies $\alpha(\mu * \nu) = \alpha\mu * \nu = \mu * \alpha\nu$ for all $\alpha \in \mathbb{C}$. (Thus, M, with multiplication defined by $\mu \cdot \nu = \mu * \nu$, is a commutative algebra over $\mathbb{C}$. Since we already know that M is a Banach space, the inequality in (1) now means that M is in fact a complex Banach algebra.)

(4) Let $\mu, \nu \in M$. Prove that for every Borel set $E \subset \mathbb{R}$,

$$(\mu * \nu)(E) = \int_{\mathbb{R}} \mu\big(\{x - t \colon x \in E\}\big) \, d\nu(t).$$

(5) Say that $\mu \in M$ is *discrete* if there is a countable set $S \subset \mathbb{R}$ such that $\mathbb{R} \setminus S$ is μ-null, and say that $\mu \in M$ is *continuous* if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$. Prove that if $\mu, \nu \in M$ are both discrete, then $\mu * \nu$ is discrete. Prove that if $\mu, \nu \in M$ and μ is continuous, then $\mu * \nu$ is continuous.

(6) As usual, let m be Lebesgue measure on $\mathbb{R}$. (Note that $m \notin M$.) Prove that if $\mu, \nu \in M$ and $\mu \ll m$, then $\mu * \nu \ll m$.

(7) Prove that the discrete measures form a closed subalgebra of M and that the continuous measures form a closed ideal in M.

(8) Recall that if λ is a (nonnegative) measure on $(X, \mathcal{M})$ and $f\colon X \to \mathbb{C}$ is integrable or nonnegative, then $f \cdot \lambda$ is the (complex or nonnegative) measure on X defined by $(f \cdot \lambda)(E) = \int_E f \, d\lambda$. Prove that $f \mapsto f \cdot m$ defines an isometric linear map which preserves the multiplication given by convolution from $L^1(\mathbb{R})$ to $\big\{\mu \in M \colon \mu \ll m\big\}$. Use this fact to prove that the operation $(f, g) \mapsto f * g$ makes $L^1(\mathbb{R})$ a commutative Banach algebra. Also prove that $\big\{\mu \in M \colon \mu \ll m\big\}$ is a closed ideal in M.

(9) Prove that the algebra M is unital, but that $L^1(\mathbb{R})$ is not unital.

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You may use without proof the obvious analog of Fubini's Theorem for complex measures. It is proved by writing each complex measure as a linear combination of four nonnegative measures, but the product then has 16 terms. Alternatively

Hint for Part (9). The easiest way to show that $L^1(\mathbb{R})$ is not unital is to see what would happen to an identity element under the Fourier transform map $f \mapsto \widehat{f}$. Feel free to use that, even if we haven't yet discussed Fourier transforms.

A more useful method, because it generalizes better, to to prove (using the definitions in a previous homework problem) a sufficient special case of the fact that if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then $f * g$ is continuous.

Almost all of this works for any locally compact Hausdorff group G in place of $\mathbb{R}$, and with a (left) Haar measure μ in place of m. The algebra is then called $M(G)$, the *measure algebra* of G. The algebra $L^1(G)$ is unital if and only if G has the discrete topology; in this case, the Haar measure can be taken to be counting measure, and $M(G) = L^1(G, \mu)$. If G is not commutative, one must be a little more careful with the formulas. The outcome is that commutativity of G, of $M(G)$, and of $L^1(G)$, are all equivalent.