MATH 617 (WINTER 2024, PHILLIPS): HOMEWORK 7

Conventions on measures: m is ordinary Lebesgue measure, $\overline{m} = (2\pi)^{-1/2}m$, and in expressions of the form $\int_{\mathbb{R}} f(x)\, dx$, ordinary Lebesgue measure is assumed.

Problem 1 (Rudin, Chapter 9, Problem 1)**.** Let $f \in L^1(\mathbb{R}, \overline{m})$, and suppose that $f \neq 0$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that $\left|\widehat{f}(y)\right| < \widehat{f}(0)$ for all $y \in \mathbb{R} \setminus \{0\}$.

The problem in Rudin is not clearly stated. It is likely to be interpreted as assuming the stronger hypothesis $f(x) > 0$ for all $x \in \mathbb{R}$. The stronger assumption doesn't help with the proof.

I have restated the next problem in labelled parts for convenience. It counts as three ordinary problems.

Problem 2 (Rudin, Chapter 9, Problem 2)**.**

(1) Compute the Fourier transform of the characteristic function of an interval.

(2) For $n \in \mathbb{Z}_{>0}$ let g_n be the characteristic function of $[-n, n]$, and let h be the characteristic function of $[-1, 1]$. Compute $g_n * h$ explicitly. (It is piecewise linear.)

(3) For $x \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{Z}_{>0}$, set

$$f_n(x) = \frac{\sin(x)\sin(nx)}{x^2}.$$

Prove that there is a constant c such that $g_n * h$ is the Fourier transform of cf_n.

(4) Let f_n be as in part (3). Prove that $\lim_{n\to\infty} \|f_n\|_1 = \infty$.

(5) Conclude that $\left\{\widehat{f}\colon f \in L^1(\mathbb{R})\right\}$ is a *proper* subset of $C_0(\mathbb{R})$.

(6) Prove that $\left\{\widehat{f}\colon f \in L^1(\mathbb{R})\right\}$ is dense $C_0(\mathbb{R})$.

Problem 3 (Rudin, Chapter 9, Problem 8)**.** Let $p \in [1, \infty]$, and let $q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $f * g$ is uniformly continuous. If $1 < p < \infty$, prove that $f * g \in C_0(\mathbb{R})$. Show by example that $f * g$ need not be in $C_0(\mathbb{R})$ when $p = 1$.

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