MATH 617 (WINTER 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 8

Conventions on measures: m is ordinary Lebesgue measure, $\overline{m} = (2\pi)^{-1/2}m$, and in expressions of the form $\int_{\mathbb{R}} f(x)\,dx$, ordinary Lebesgue measure is assumed.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

Problem 1 (Rudin, Chapter 9, Problem 4)**.** Give an explicit example of a function $f \in L^2(\mathbb{R})$ such that $f \notin L^1(\mathbb{R})$ but $\widehat{f} \in L^1(\mathbb{R})$. Under what circumstances can this happen?

Solution. We answer the second part first: f is such a function if and only if $\widehat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ but $\widehat{f}$ is not the Fourier transform of a function in $L^1(\mathbb{R})$. In particular (although this is not the most general possibility), this will happen for any $f \in L^2(\mathbb{R})$ such that $\widehat{f} \in L^1(\mathbb{R})$ but $\widehat{f} \notin C_0(\mathbb{R})$.

To get an example of such a function, choose $g \in [L^2(\mathbb{R}) \cap L^1(\mathbb{R})] \setminus C_0(\mathbb{R})$, and let f be the inverse Fourier transform of g. (Taking $f = \widehat{g}$ will also work, since then $\widehat{f}(t) = g(-t)$ for all t. One sees this immediately by comparing the formulas for the Fourier transform and the inverse Fourier transform for, say, functions in the Schwartz space, and using density of this space in $L^2(\mathbb{R})$.)

For a specific example, take $g = \chi_{[-1,\,1]} \in L^2(\mathbb{R})$. Then set

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} g(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{itx}\,dt = \sqrt{\frac{2}{\pi}} \left(\frac{\sin(x)}{x} \right).$$

The Fourier inversion formula for $L^2(\mathbb{R})$ implies that $\widehat{f} = g$, which is in $L^1(\mathbb{R})$ as desired. One can show directly that $f \notin L^1(\mathbb{R})$, or simply observe that $\widehat{f}$ is not continuous. ☐

Problem 2 (Rudin, Chapter 9, Problem 5)**.** Let $f \in L^1(\mathbb{R})$, and suppose that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| t\widehat{f}(t) \right| dt$$

is finite. Prove that there exists a function $g\colon \mathbb{R} \to \mathbb{C}$ such that $f(x) = g(x)$ for almost all $x \in \mathbb{R}$ and such that for all $x \in \mathbb{R}$ we have

$$g'(x) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} t\widehat{f}(t) e^{ixt}\,dt.$$

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Solution. We claim that $\widehat{f} \in L^1(\mathbb{R})$. To see this, use the fact that $\widehat{f}$ is continuous to find M such that $\left|\widehat{f}(t)\right| \leq M$ for all $t \in [-1,1]$. Set $E = \mathbb{R} \setminus [-1,1]$.

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left|\widehat{f}(t)\right| dt &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left|\widehat{f}(t)\right| dt + \frac{1}{\sqrt{2\pi}} \int_{E} \left|\widehat{f}(t)\right| dt \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} M\, dt + \frac{1}{\sqrt{2\pi}} \int_{E} \left|t\widehat{f}(t)\right| dt \\ &\leq 2M + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left|t\widehat{f}(t)\right| dt < \infty.\end{aligned}$$

This proves the claim.

Define $h(t) = \widehat{f}(t)$ and $k(t) = -it\widehat{f}(t)$ for $t \in \mathbb{R}$. Then $h, k \in L^1(\mathbb{R})$. So Theorem 9.2(f) of Rudin implies that $\widehat{h}$ is differentiable and $\left(\widehat{h}\right)'(x) = \widehat{k}(x)$ for all $x \in \mathbb{R}$. Since f and $h = \widehat{f}$ are in $L^1(\mathbb{R})$, the Fourier inversion theorem applies and shows that for almost all $x \in \mathbb{R}$ we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(t) e^{ixt}\, dt = \widehat{h}(-x).$$

Therefore the function $g(x) = \widehat{h}(-x)$ agrees with f for almost all $x \in \mathbb{R}$, and satisfies

$$g'(x) = -\left(\widehat{h}\right)'(-x) = -\widehat{k}(-x) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} t\widehat{f}(t) e^{ixt}\, dt.$$

This completes the proof. □

Alternate solution. For $x \in \mathbb{R}$, the function $t \mapsto t\widehat{f}(t)e^{itx}$ is integrable because $\left|t\widehat{f}(t)e^{itx}\right| = \left|t\widehat{f}(t)\right|$. So we can define

$$h(x) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} t\widehat{f}(t) e^{itx}\, dt.$$

We claim that h is continuous. Let $x \in \mathbb{R}$ and let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$. The functions $k(t) = t\widehat{f}(t)e^{itx}$ and $k_n(t) = t\widehat{f}(t)e^{itx_n}$ for $n \in \mathbb{Z}_{>0}$ satisfy $\lim_{n\to\infty} k_n(t) = k(t)$ for all $t \in \mathbb{R}$. We may therefore apply the Dominated Convergence Theorem, with the bounding function being $t \mapsto \left|t\widehat{f}(t)\right|$, to conclude that

$$h(x) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} k(t)\, dt = \lim_{n\to\infty} \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} k_n(t)\, dt = \lim_{n\to\infty} h_n(x).$$

The claim follows.

For every bounded interval $I \subset \mathbb{R}$, the function $(x,s) \mapsto s\widehat{f}(s)e^{isx}$ is integrable on $I \times \mathbb{R}$. For $t \in \mathbb{R}$, we can therefore apply Fubini's Theorem for integrable functions at the second step (even if $t < 0$, using $I = [t, 0]$) to get

$$\begin{aligned}\int_0^t h(x)\, dx &= \frac{i}{\sqrt{2\pi}} \int_0^t \left(\int_{\mathbb{R}} s\widehat{f}(s) e^{isx}\, ds \right) dx = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_0^t s\widehat{f}(s) e^{isx}\, dx \right) ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(s)(e^{ist} - 1)\, ds.\end{aligned}$$

We next claim that $\widehat{f} \in L^1(\mathbb{R})$. The proof is the same as at the beginning of the first solution.

Given the claim, and using $f \in L^1(\mathbb{R})$, the Fourier Inversion Theorem implies that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(s)(e^{ist} - 1)\, ds = f(x) - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(s)\, ds$$

for almost every $x \in \mathbb{R}$. Define

$$C = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(s)\, ds$$

and for $x \in \mathbb{R}$ define

$$g(x) = \int_0^x h(y)\, dy - C.$$

Thus $g(x) = f(x)$ for almost every $x \in \mathbb{R}$. Since h is continuous, the Fundamental Theorem of Calculus gives

$$g'(x) = h(x) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} t\widehat{f}(t)e^{itx}\, dt$$

for all $x \in \mathbb{R}$. □

Problem 3 (Rudin, Chapter 9, Problem 7)**.** Let S be the set of all C^∞ functions $f\colon \mathbb{R} \to \mathbb{C}$ such that for all $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\sup_{x\in\mathbb{R}} \left| x^n f^{(m)}(x) \right| < \infty. \tag{1}$$

Prove that $f \mapsto \widehat{f}$ is a bijection from S to S. Give examples of nonzero elements of S.

Comments: The space S is a topological vector space with topology given by the seminorms implicit in (1) for $m, n \in \mathbb{Z}_{\geq 0}$, and the map $f \mapsto \widehat{f}$ is a homeomorphism. Also, one gets the same topology with different choices of seminorms. For example, one could use the family of seminorms given by

$$\|f\|_{m,n} = \left(\int_{\mathbb{R}} (1 + x^{2n})^{1/2} |f^{(m)}(x)|^2 \, d\overline{m}(x) \right)^{1/2}$$

for $m, n \in \mathbb{Z}_{\geq 0}$, or an L^p version of these seminorms for any $p \in [1, \infty)$. The reason is that arbitrarily large powers of x appear. For example, if f is continuous and $x \mapsto x^2 f(x)$ is bounded, then f is an L^1 function on $\mathbb{R}$.

It is convenient to break the solution in several lemmas. We also let F denote the Fourier transform, as a map from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$.

Lemma 4. We have $S \subset L^1(\mathbb{R}) \cap C_0(\mathbb{R})$.

Proof. Let $f \in S$. Since $\sup_{x\in\mathbb{R}} |xf(x)| < \infty$ and $\lim_{x\to\infty} |x| = \infty$, it is immediate that $\lim_{x\to\infty} f(x) = 0$. Similarly $\lim_{x\to-\infty} f(x) = 0$. Thus $f \in C_0(\mathbb{R})$.

Now let

$$M_1 = \sup_{x\in\mathbb{R}} |x^2 f(x)| \quad \text{and} \quad M_2 = \sup_{x\in\mathbb{R}} |f(x)|.$$

Define $b(x) = \min(M_1 x^{-2},\, M_2)$. Then $b \in L^1(\mathbb{R})$ and $|f| \leq b$, so $f \in L^1(\mathbb{R})$. □

Lemma 5. The space S is a vector space, and if $f \in S$ then the functions f' and $x \mapsto xf(x)$ are in S.

Proof. It is obvious that S is a vector space. It is also obvious that if $f \in S$ then $f' \in S$.

Now let $f \in S$, and set $g(x) = xf(x)$. An induction argument shows that

$$g^{(m)}(x) = xf^{(m)}(x) + mf^{(m-1)}(x)$$

for every $m \in \mathbb{Z}_{>0}$ and $x \in \mathbb{R}$. Therefore

$$\sup_{x\in\mathbb{R}} |x^n g^{(m)}(x)| \leq \sup_{x\in\mathbb{R}} |x^{n+1} f^{(m)}(x)| + \sup_{x\in\mathbb{R}} |mx^n f^{(m-1)}(x)| < \infty.$$

Thus $g \in S$. $\square$

The next lemma is the reverse of Theorem 9.2(f) of Rudin.

Lemma 6. Let $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, and suppose that also f' and $x \mapsto xf(x)$ are in $L^1(\mathbb{R}) \cap C_0(\mathbb{R})$. Then $F(f')(t) = itF(f)(t)$ for all $t \in \mathbb{R}$.

Proof. We have

$$F(f')(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} f'(x)\, dx.$$

Integrate by parts, using

$$\lim_{x\to\infty} e^{-itx} f(x) = \lim_{x\to-\infty} e^{-itx} f(x) = 0,$$

and the fact that $x \mapsto xf(x)$ is in $L^1(\mathbb{R})$. $\square$

Proposition 7. If $f \in S$ then $\widehat{f} \in S$.

Proof. For $m, n \in \mathbb{Z}_{\geq 0}$ set $g_{m,n}(t) = t^n (F(f))^{(m)}(t)$, set $f_{m,0}(x) = (-ix)^m f(x)$, and set $f_{m,n}(x) = (-i)^n f_{m,0}^{(n)}$. It suffices to prove that $g_{m,n} \in C_0(\mathbb{R})$ for all $m, n \in \mathbb{Z}_{\geq 0}$. By repeated application of Theorem 9.2(f) of Rudin, using Lemmas 4 and 5 to verify its hypotheses at each step, we see that $g_{m,0} = F(f_{m,0})$ for every $m \geq 0$. Repeated application of Lemma 6, justified the same way, then shows that $g_{m,n} = F(f_{m,n})$ for every $m, n \geq 0$. Now $g_{m,n} \in C_0(\mathbb{R})$ because it is the Fourier transform of the L^1 function $f_{m,n}$. $\square$

Proposition 8. If $f \in S$ then there is $g \in S$ such that $F(g) = f$.

Proof. For every $h \in S$ we have both $h \in L^1(\mathbb{R})$ and $F(h) \in L^1(\mathbb{R})$. Therefore the Fourier inversion theorem applies, and shows that

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} F(h)(t)\, dt = F(F(h))(-x)$$

for all $x \in \mathbb{R}$. (The second step follows by inspection.) Repeating, we get $h = F^4(h)$. Three applications of Proposition 7 show that $F^3(f) \in S$, so $g = F^3(f)$ will work. $\square$

Alternate proof of Proposition 8. Using the same method as in the first proof of Proposition 8, we show that for every $h \in S$ we have $h(x) = F(F(h))(-x)$ for all $x \in \mathbb{R}$. Now set $g(x) = F(f)(-x)$ for all $x \in X$. Proposition 7 shows that $F(f) \in S$. It is easy to check that if $h \in S$ then so is the function $k(x) = h(-x)$ for $x \in \mathbb{R}$, and moreover (details omitted) that $F(k)(t) = F(f)(-t)$ for all $t \in \mathbb{R}$. Therefore $g \in S$ and $F(g)(t) = F^2(f)(-x) = f(x)$ for all $x \in \mathbb{R}$. $\square$

It is clear that $F|_S$ is injective, because F itself is injective.

Now we give some examples of elements of S. We give proofs for several of them below. You need to give, with proof, a nonzero example.

(1) The zero function is trivially in S.
(2) $f(x) = \exp(-x^2)$ defines an element of S. Moreover, if p is any polynomial, or any trigonometric polynomial, then $g(x) = p(x)\exp(-x^2)$ defines an element of S. One can use other functions too. For example,

$$g(x) = \exp(-x^2)\sqrt{1+x^2} \quad \text{and} \quad h(x) = \frac{\exp(-x^2)}{\sqrt{1+x^2}}$$

both define elements of S.

Proving that $g(x) = p(x)\exp(-x^2)$ defines an element of S amounts to showing that $x \mapsto x^n \exp(-x^2)$ is bounded for every $n \in \mathbb{Z}_{\geq 0}$. This can be done using standard estimates, but one clever way to do it is to use methods of elementary calculus to show that the maximum value of $|x^n \exp(-x^2)|$ occurs at $\pm\sqrt{n/2}$.
(3) More generally, all that was said in (2) is also true if the function $\exp(-x^2)$ is replaced by $\exp(-x^{2n})$ for any $n \in \mathbb{Z}_{>0}$.
(4) Any C^∞ function with compact support is in S. Here is an explicit example. Fix $a, b \in \mathbb{R}$ with $a < b$, and take

$$f(x) = \begin{cases} 0 & x \leq a \\ \exp\left(-\frac{1}{(x-a)(b-x)}\right) & a < x < b \\ 0 & b \leq x. \end{cases}$$

We give a straightforward proof that if p is any polynomial function, then $x \mapsto p(x)\exp(-x^2)$ defines an element of S. The proof is just as easy with $\exp(-x^{2n})$ in place of $\exp(-x^2)$, so we give it in that generality.

Proposition 9. Let $r \in \mathbb{Z}_{>0}$ and let $p\colon \mathbb{R} \to \mathbb{C}$ be any polynomial function. Define $f(x) = p(x)\exp(-x^{2r})$ for $x \in \mathbb{R}$. Then $f \in S$.

Proof. For $m \in \mathbb{Z}_{\geq 0}$, define a polynomial function $q_n\colon \mathbb{R} \to \mathbb{C}$ inductively by $q_0(x) = p(x)$ for all $x \in \mathbb{R}$ and $q_{m+1}(x) = q_m'(x) - 2rx^{2r-1}q_m(x)$ for all $x \in \mathbb{R}$. An induction argument shows that $f^{(m)}(x) = q_m(x)\exp(-x^{2r})$ for $m \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$. Let d_m be the degree of q_m.

Let $m, n \in \mathbb{Z}_{\geq 0}$, and choose $l \in \mathbb{Z}_{\geq 0}$ such that $2rl \geq d_m + n$. Let M be the sum of the absolute values of the coefficients of q_m. Then $|q_m(x)| \leq M|x|^{d_m}$ when $|x| \geq 1$ and $|q_m(x)| \leq M$ when $|x| \leq 1$, so

$$|q_m(x)| \leq M\max(|x|^{d_m}, 1) = M\max(|x|, 1)^{d_m}$$

for all $x \in \mathbb{R}$. For $x \in \mathbb{R}$, we have

$$\exp(x^{2r}) = \sum_{k=0}^{\infty} \frac{x^2rk}{k!} \geq \max\left(1, \frac{x^{2rl}}{l!}\right) \geq \left(\frac{1}{l!}\right)\max(1, x^{2rl}) = \left(\frac{1}{l!}\right)\max(|x|, 1)^{2rl},$$

so that

$$\begin{aligned}\sup_{x\in\mathbb{R}}|x^n f^{(m)}(x)| &= \sup_{x\in\mathbb{R}}\frac{|x|^n|q_m(x)|}{\exp(x^{2r})} \leq \sup_{x\in\mathbb{R}}\frac{M|x|^n\max(|x|,1)^{d_m}}{\left(\frac{1}{l!}\right)\max(|x|,1)^{2rl}}\\ &\leq \sup_{x\in\mathbb{R}}\frac{Ml!\max(|x|,1)^{d_m+n}}{(|x|,1)^{2rl}} = Ml!\sup_{x\in\mathbb{R}}\max(|x|,1)^{d_m+n-2rl}.\end{aligned}$$

Since $d_m+n-2rl \leq 0$, this last expression is at most $Ml!$. Thus $\sup_{x\in\mathbb{R}}|x^n f^{(m)}(x)|$ is finite. Since $m, n \in \mathbb{Z}_{\geq 0}$ are arbitrary, we have shown that $f \in S$. □

Problem 10 (Rudin, Chapter 9, Problem 9)**.** Let $p \in [1,\infty)$, let $f \in L^p(\mathbb{R})$, and define $g\colon \mathbb{R} \to \mathbb{C}$ by

$$g(x) = \int_x^{x+1} f(t)\,dt.$$

Prove that $g \in C_0(\mathbb{R})$. What can you say if $f \in L^\infty(\mathbb{R})$?

Solution. We prove that g is continuous. Let $q \in (1,\infty]$ satisfy $\frac{1}{p}+\frac{1}{q}=1$. We can write

$$\begin{aligned}g(x) &= \int_{\mathbb{R}}\chi_{[x,\,x+1]}(t)f(t)\,dt = \int_{\mathbb{R}}\chi_{[0,1]}(t-x)f(t)\,dt\\ &= \int_{\mathbb{R}}\chi_{[0,1]}(t)f(t+x)\,dt = \int_{\mathbb{R}}\chi_{[0,1]}(t)f_{-x}(t)\,dt.\end{aligned}$$

Theorem 9.5 of Rudin implies that $x \mapsto f_{-x}$ is continuous from $\mathbb{R}$ to $L^p(\mathbb{R})$, and $\chi_{[0,1]} \in L^q(\mathbb{R})$, so

$$x \mapsto \int_{\mathbb{R}}\chi_{[0,1]}(t)f_{-x}(t)\,dt$$

is continuous.

We now claim that

$$\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = 0.$$

Let $\varepsilon > 0$. Since $\int_{\mathbb{R}}|f(t)|^p\,dt < \infty$, it follows that

$$\sum_{n=-\infty}^{\infty}\int_n^{n+1}|f(t)|^p\,dt < \infty.$$

Therefore there exists $N \in \mathbb{Z}_{>0}$ such that whenever $n \in \mathbb{Z}$ satisfies $|n| > N$, we have

$$\int_n^{n+1}|f(t)|^p\,dt < \frac{\varepsilon^p}{2^p}.$$

Now let $x \in \mathbb{R}$ satisfy $|x| > N+3$. Then there exists $n \in \mathbb{Z}$ such that $|n| > N$, $|n+1| > N$, and $[x,\,x+1] \subset [n,\,n+2]$. Therefore, using Hölder's inequality at the third step, we have

$$\begin{aligned}
|g(x)| &\le \int_n^{n+1} |f(t)|\,dt + \int_{n+1}^{n+2} |f(t)|\,dt \\
&= \int_{\mathbb{R}} \chi_{[n,\,n+1]}(t)|f(t)|\,dt + \int_{\mathbb{R}} \chi_{[n+1,\,n+2]}(t)|f(t)|\,dt \\
&\le \|\chi_{[n,\,n+1]}\|_q \left(\int_n^{n+1} |f(t)|^p\,dt\right)^{1/p} + \|\chi_{[n+1,\,n+2]}\|_q \left(\int_{n+1}^{n+2} |f(t)|^p\,dt\right)^{1/p} \\
&< \left(\frac{\varepsilon^p}{2^p}\right)^{1/p} + \left(\frac{\varepsilon^p}{2^p}\right)^{1/p} = \varepsilon.
\end{aligned}$$

This proves the claim, and we conclude that $g \in C_0(\mathbb{R})$.

Suppose now that $f \in L^\infty(\mathbb{R})$. We claim that g is bounded and uniformly continuous, but need not vanish at ∞.

For the last part, taking $f(x) = 1$ for all $x \in \mathbb{R}$ gives $g(x) = 1$ for all $x \in \mathbb{R}$. Boundedness follows from the estimate

$$|g(x)| \le \int_x^{x+1} |f(t)|\,dt \le \|f\|_\infty.$$

To prove uniform continuity, we write

$$g(x) = \int_{\mathbb{R}} \chi_{[x,\,x+1]}(t)f(t)\,dt = \int_{\mathbb{R}} \chi_{[0,1]}(t-x)f(t)\,dt = \int_{\mathbb{R}} (\chi_{[0,1]})_x(t)f(t)\,dt.$$

Theorem 9.5 of Rudin implies that $x \mapsto (\chi_{[0,1]})_x$ is continuous from $\mathbb{R}$ to $L^1(\mathbb{R})$, and $f \in L^\infty(\mathbb{R})$, so

$$x \mapsto \int_{\mathbb{R}} (\chi_{[0,1]})_x(t)f(t)\,dt$$

is uniformly continuous. (This argument actually works whenever $p \in (1, \infty]$, but not for $p = 1$.) $\square$