MATH 617 (WINTER 2024, PHILLIPS): HOMEWORK 9

Conventions on measures: m is ordinary Lebesgue measure, $\overline{m} = (2\pi)^{-1/2}m$, and in expressions of the form $\int_{\mathbb{R}} f(x)\,dx$, ordinary Lebesgue measure is assumed.

Problem 1. Let X be a locally compact σ-compact Hausdorff space. Prove that there is a sequence $(g_n)_{n \in \mathbb{Z}_{>0}}$ in $C_0(X)$ consisting of functions with compact support and with values in $[0,1]$ such that for every $f \in C_0(X)$ we have $\lim_{n\to\infty} \|g_n f - f\|_\infty = 0$.

If X is not σ-compact, one needs a net instead of a sequence. You will need to prove that there are compact subsets $K_1, K_2, \ldots \subset X$ such that

$$K_1 \subset \operatorname{int}(K_2) \subset K_2 \subset \operatorname{int}(K_3) \subset K_3 \subset \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} K_n = X.$$

(I didn't find this explicitly in Rudin's book, but maybe I didn't look in the right place.)

Problem 2. Give a "direct" proof of the following part of Theorem 9.6 of Rudin's book: if $f \in L^1$ then $\widehat{f} \in C_0(\mathbb{R})$. That is, prove this first when f is the characteristic function of a bounded interval, use this result and approximation to prove $\widehat{f} \in C_0(\mathbb{R})$ when $f \in C_c(\mathbb{R})$, and then use approximation to prove $\widehat{f} \in C_0(\mathbb{R})$ for general $f \in L^1$.

You will need $\left|\widehat{f}(t)\right| \leq \|f\|_1$ for all $t \in \mathbb{R}$. This proof takes longer than Rudin's proof, but the methods are useful much more generally, and the first step explains why the result is even true.

Problem 3 counts as two ordinary problems.

Problem 3 (Rudin, Chapter 9, Problem 13abc)**.** For $c \in (0,\infty)$ define $f_c \colon \mathbb{R} \to \mathbb{C}$ by $f_c(x) = \exp(-cx^2)$ for $x \in \mathbb{R}$.

(1) Compute $\widehat{f_c}$.
(2) Prove that there exists a unique $c \in (0,\infty)$ such that $\widehat{f_c} = f_c$.
(3) Let $a, b \in (0,\infty)$. Prove that there exist γ and c such that $f_a * f_b = \gamma f_c$, and find explicit formulas for γ and c in terms of a and b.

You may take as known the result that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. (This is proved by writing the square of the integral as $\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy$ and computing it with polar coordinates in $\mathbb{R}^2$.)

Hint for part (1). One method (not the only possible method) is to let $g = \widehat{f_c}$, and use integration by parts to get $2cg'(t) + tg(t) = 0$ for all $t \in \mathbb{R}$. If you use this method, you will need to prove (directly, or by citing theorems) that this equation, together with other information you have, determines g uniquely.

Problem 4. Let X be a topological space. Prove that X is Hausdorff if and only if every net in X has an most one limit. Give an example to show that this result fails if one uses sequences in place of nets.

Hint for the example. Use a modification of the set of ordinals less than or equal to the first uncountable ordinal.

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