MATH 617 FINAL EXAM SOLUTIONS (WINTER 2007)

1. (10 points) State the Radon-Nikodym Theorem and the Lebesgue Decomposition Theorem.

(These are usually considered two separate theorems. They were combined in the book, and you can give the combined statement if you like.)

Solution: For the combined statement, see Theorem 6.10 of Rudin's book. Here are the separate statements:

Radon-Nikodym Theorem. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, with μ a positive measure. Let ν be a complex measure defined on the σ-algebra $\mathcal{M}$, and assume ν is absolutely continuous with respect to μ. Then there is a unique $h \in L^1(\mu)$ (the *Radon-Nikodym derivative* of ν with respect to μ) such that $\nu(E) = \int_E h \, d\mu$ for all $E \in \mathcal{M}$. (In terms of functions, the uniqueness assertion is that any other function with this property must be equal to h almost everywhere $[\mu]$.)

Lebesgue Decomposition Theorem. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, with μ a positive measure. Let ν be a complex measure defined on the σ-algebra $\mathcal{M}$. Then there is a unique pair (λ, φ) of complex measures defined on $\mathcal{M}$ such that $\nu = \lambda + \rho$, λ is absolutely continuous with respect to μ, and ρ is mutually singular with respect to μ.

2. (a) (10 points) State Fubini's Theorem.

Fubini's Theorem. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Let $f \colon X \times Y \to [0, \infty]$ be measurable with respect to $\mathcal{M} \times \mathcal{N}$, or let $f \colon X \times Y \to \mathbb{C}$ be integrable with respect to $\mu \times \nu$. Then:

(1) For every $y \in Y$, the function $x \mapsto f(x, y)$ is measurable with respect to $\mathcal{M}$, and, in the second case, for almost every y with respect to ν, this function is integrable with respect to μ.

(2) The function $y \mapsto \int_X f(x, y) \, d\mu(x)$ is measurable with respect to ν, and in the second case this function is integrable with respect to ν.

(3) $\int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\nu(y) = \int_{X \times Y} f \, d(\mu \times \nu)$.

The version above compares only one iterated integral with the integral with respect to the product measure. It is perfectly acceptable to compare the other one, or both. I will also accept the version for the completion of the product of complete measures, provided everything is correctly stated.

(b) (35 points) Define $f \colon (0, 1) \times (0, 1) \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 0 & 0 < x \leq y < 1 \\ x^{-3/2} \sin(1/(xy)) & 0 < y < x < 1. \end{cases}$$

Prove that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = \int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx.$$

Solution: We will use Fubini's Theorem.

We first show that f is measurable. The function $g(x,y) = x^{-3/2}\sin(1/(xy))$ is continuous on $(0,1) \times (0,1)$, hence measurable. The set $T = \{(x,y) \in \mathbb{R}^2 \colon 0 < y < x < 1\}$ is open, so χ_T is measurable. Therefore $\chi_T g$ is measurable, but this function is equal to f.

Next, we show that f is integrable on $(0,1) \times (0,1)$. Set Define $f \colon (0,1) \times (0,1) \to \mathbb{R}$ by

$$h(x,y) = \begin{cases} 0 & 0 < x \leq y < 1 \\ x^{-3/2} & 0 < y < x < 1. \end{cases}$$

This function is measurable by the same reasoning as for f, and it is nonnegative. It is useful because $|f| \leq h$.

Let m be Lebesgue measure on $(0,1)$. Applying Fubini's Theorem for nonnegative functions, we get

$$\int_{(0,1)\times(0,1)} h\,d(m \times m)(x,y) = \int_0^1 \int_0^x x^{-3/2}\,dy\,dx = \int_0^1 x^{-1/2}\,dx = 2.$$

So h is integrable. Therefore f is integrable.

Now Fubini's Theorem for integrable functions implies that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = \int_{(0,1)\times(0,1)} f\,d(m \times m) = \int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx,$$

as desired. ∎

Remark: Since f takes both positive and negative values, one may not apply Fubini's Theorem without checking integrability first. Much of the credit is therefore for doing this step.

3. (a) (5 points) State the definition of a Lebesgue point.

Solution: This is Definition 7.6 of Rudin's book. It is repeated here for reference in the next part of the problem.

Definition. Let $f \in L^1(\mathbb{R}^d)$, and let $x \in \mathbb{R}^d$. Then x is a *Lebesgue point* of f is

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f - f(x)|\,dm = 0.$$

Here m is Lebesgue measure on $\mathbb{R}^d$, and

$$B_r(x) = \{y \in \mathbb{R}^d \colon \|y - x\|_2 < r\}.$$

(b) (35 points) Let E be the subset of $\mathbb{R}^2$ given by

$$E = \left\{ (x, y) \in \mathbb{R}^2 \colon -1 \leq x \leq 1, \, -1 \leq y \leq \sqrt{|x|} \right\}.$$

Consider the points in $\mathbb{R}^2$:

$$(-7, \, -17), \quad (0, \, -1), \quad \left(0, \, -\tfrac{1}{2}\right), \quad (0, 0), \quad \left(\tfrac{1}{2}, \sqrt{\tfrac{1}{2}}\right).$$

For each of the five points c listed above, determine, with proof, whether there exists a number $\lambda_c \in \mathbb{C}$ such that c is a Lebesgue point of the function

$$f(p) = \begin{cases} \lambda_c & p = c \\ \chi_E(p) & p \neq c. \end{cases}$$

Solution: Suppose $x \notin E$. (In particular, this applies to $(-7, \, -17)$.) Since E is closed, we have $B_r(x) \cap E = \varnothing$ for all sufficiently small r. (Taking $r < \operatorname{dist}(x, E)$ will do.) So x is a Lebesgue point of χ_E. In particular, for $c = (-7, \, -17)$, taking $\lambda_c = 0 = \chi_E(c)$ will work.

Next consider $x \in \operatorname{int}(E)$. (In particular, this applies to $\left(0, \, -\frac{1}{2}\right)$.) Then $B_r(x) \cap E = B_r(x)$ for all sufficiently small r. (Taking $r < \operatorname{dist}(x, \, \mathbb{R}^2 \setminus E)$ will do.) So again x is a Lebesgue point of χ_E. In particular, for $c = \left(0, \, -\frac{1}{2}\right)$, taking $\lambda_c = 0 = \chi_E(c)$ will work.

Next consider $c = (0, \, -1)$. For all $r \leq 1$, we have

$$m(B_r(c) \cap E) = \tfrac{1}{2} m(B_r(c)) \quad \text{and} \quad m(B_r(c) \cap (\mathbb{R}^2 \setminus E)) = \tfrac{1}{2} m(B_r(c)).$$

If we redefine $f(c)$ to be λ, then for $r < 1$ we have

$$\begin{aligned} \frac{1}{m(B_r(c))} \int_{B_r(c)} |f - \lambda| \, dm &= \frac{1}{m(B_r(c))} \left(\int_{B_r(c) \cap E} |\lambda| \, dm + \int_{B_r(c) \cap (\mathbb{R}^2 \setminus E)} |1 - \lambda| \, dm \right) \\ &= \tfrac{1}{2} |\lambda| + \tfrac{1}{2} |1 - \lambda|. \end{aligned}$$

This does not approach zero for any value of λ, so λ_c does not exist.

Next let $c = \left(\frac{1}{2}, \sqrt{\frac{1}{2}}\right)$. It is possible to show, using differentiability of the function $f(a) = \sqrt{a}$ at t, that

$$\lim_{r \to 0} \frac{1}{m(B_r(c))} \int_{B_r(c)} |f - \lambda| \, dm = \tfrac{1}{2} |\lambda| + \tfrac{1}{2} |1 - \lambda|,$$

just as for $c = (0, \, -1)$. However, there is an easier approach. Recall that if (S_n) is a sequence of sets which shrinks nicely to c, and if c is a Lebesgue point of f, then

$$\lim_{n \to \infty} \frac{1}{m(S_n)} \int_{S_n} |f - f(c)| \, dm = 0.$$

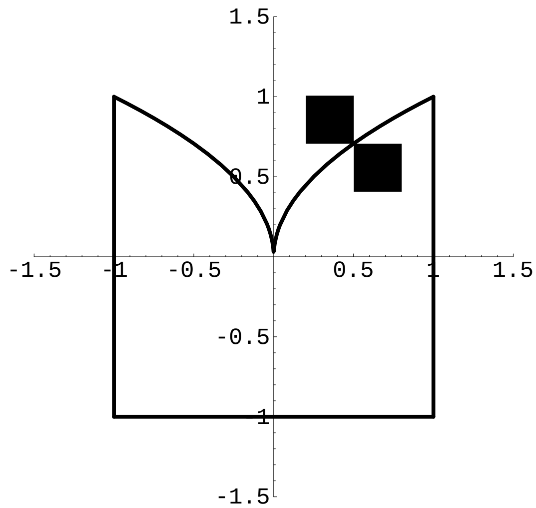
This implies

$$\lim_{n \to \infty} \frac{m(S_n \cap E)}{m(S_n)} = f(c) = \lambda_c.$$

Now set $t = \frac{1}{2}$, and choose

$$S_n = \left(\left[t,\, t + \tfrac{1}{n}\right] \times \left[\sqrt{t} - \tfrac{1}{n},\, \sqrt{t}\right] \right) \quad \text{and} \quad T_n = \left(\left[t - \tfrac{1}{n},\, t\right] \times \left[\sqrt{t},\, \sqrt{t} + \tfrac{1}{n}\right] \right).$$

Here is a picture, in which S_n is the lower right black square and T_n is the upper left black square:



These sets are easily seen to shrink nicely to x. If $n \geq 2$, then the first of the two products in this definition is contained in E, and the second is disjoint from E (except for the point $x = (t, \sqrt{t})$ itself). Therefore

$$\lim_{n\to\infty} \frac{m(S_n \cap E)}{m(S_n)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{m(T_n \cap E)}{m(T_n)} = 0.$$

Since we can't have both $\lambda_c = 0$ and $\lambda_c = 1$, it follows that λ_c does not exist.

Finally, we consider the point $c = (0, 0)$. For $0 < r < 1$, one can check that $B_r(x) \cap (\mathbb{R}^2 \setminus E)$ is contained in the sector of $B_r(0)$ between the lines through the origin with slopes

$$\frac{\sqrt{r}}{r} = \frac{1}{\sqrt{r}} \quad \text{and} \quad -\frac{\sqrt{r}}{r} = -\frac{1}{\sqrt{r}}$$

in the upper half plane, that is, the set

$$C_r = \left\{ (\rho\cos(\theta),\, \rho\sin(\theta)) \colon 0 \leq \rho < r,\, \arctan\left(\frac{1}{\sqrt{r}}\right) < \theta < \pi - \arctan\left(\frac{1}{\sqrt{r}}\right) \right\}.$$

Therefore

$$1 \geq \frac{m(B_r(0) \cap E)}{m(B_r(0))} \geq 1 - \frac{m(C_r)}{m(B_r(0))} = 1 - \frac{1}{2\pi}\left(\pi - 2\arctan\left(\frac{1}{\sqrt{r}}\right)\right).$$

Since

$$\lim_{r\to 0} \arctan\left(\frac{1}{\sqrt{r}}\right) = \frac{\pi}{2},$$

we get

$$\lim_{r\to 0} \frac{m(B_r(0)\cap E)}{m(B_r(0))} = 1.$$

This shows that $(0,0)$ is a Lebesgue point of χ_E, even though it is on the boundary of E. In particular, we can take $\lambda_c = 1$.

4. (15 points) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $M(X)$ be the Banach space of all complex measures defined on the σ-algebra $\mathcal{M}$. Let $E \subset M(X)$ be the set of all measures in $M(X)$ which are absolutely continuous with respect to μ. Prove that E is a closed subspace of $M(X)$.

Solution: The proof that E is a vector subspace is easy, and is omitted. We show that E is closed. Let (ν_n) be a sequence in E and suppose $\|\nu_n - \nu\| \to 0$. For $B \in \mathcal{M}$, we have

$$|\nu_n(B) - \nu(B)| \leq |\nu_n(B) - \nu(B)| + |\nu_n(X\setminus B) - \nu(X\setminus B)| \leq \|\nu_n - \nu\|.$$

Therefore $\nu_n(B) \to \nu(B)$ for all $B \in \mathcal{M}$.

Let $N \in \mathcal{M}$ satisfy $\mu(N) = 0$. Then

$$\nu(N) = \lim_{n\to\infty} \nu_n(N) = \lim_{n\to\infty} 0 = 0.$$

This shows that ν is absolutely continuous with respect to μ, as desired.

5. (20 points) Let E and F be Banach spaces, and let $a\colon E \to F$ be an injective bounded linear map whose range $a(E) \subset F$ is closed. Prove that there is $\delta > 0$ such that $\|a\xi\| \geq \delta\|\xi\|$ for all $\xi \in E$.

Solution: Let $F_0 = a(E)$, and let $a_0\colon E \to F_0$ be the map a with restricted codomain. Then F_0 is a closed subspace of a Banach space, hence also a Banach space. Moreover, a_0 is a bijective bounded linear map between Banach spaces. The Open Mapping Theorem therefore implies that a_0^{-1} is also bounded. It is easy to check that $\delta = \|a_0^{-1}\|^{-1}$ satisfies the required conditions.

6. (40 points) Let E be the set of bounded complex sequences $\xi = (\xi(n))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \xi(n)$ exists. For $\xi \in E$ define $\|\xi\| = \sup_{n\in\mathbb{N}} |\xi(n)|$.

Prove carefully that E is a vector space, that $\|\cdot\|$ is a norm on E, and that E is a Banach space. (A large part of the credit is for the last part.)

7. (30 points) Let E be a Banach space. Prove or disprove: If $\omega\colon E \to \mathbb{C}$ is a linear functional such that $|\omega(\xi)| < 1$ for all $\xi \in E$ with $\|\xi\| = 1$, then $\|\omega\| < 1$.

Solution: The statement is false. Here is one of the simplest examples. Take $E = l^1$, and for $\xi = (\xi_n)_{n \in \mathbb{N}} \in l^1$ set

$$\omega(\xi) = \sum_{n=1}^{\infty} \left(1 - \tfrac{1}{n}\right) \xi_n.$$

We verify the hypothesis. Let $\xi \in l^1$ satisfy $\|\xi\| = 1$. Choose n with $\xi_n \neq 0$. Then

$$|\omega(\xi)| \leq \left(1 - \tfrac{1}{n}\right) |\xi_n| + \sum_{m \neq n} \left(1 - \tfrac{1}{m}\right) |\xi_m|$$

$$\leq \sum_{m=1}^{\infty} |\xi_m| - \tfrac{1}{n}|\xi_n| = 1 - \tfrac{1}{n}|\xi_n| < 1.$$

We show that the supposed conclusion fails, by showing that, for every $\varepsilon > 0$, there is $\xi \in E$ such that $\|\xi\| = 1$ and $|\omega(\xi)| > 1 - \varepsilon$. Indeed, choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$, and take

$$\xi_m = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases}$$

Then $\|\xi\| = 1$ and $\omega(\xi) = 1 - \frac{1}{n} > 1 - \varepsilon$.