MATH 618 (SPRING 2020): FINAL EXAM SOLUTIONS

Instructions: All lemmas, claims, examples, counterexamples, etc. require proof, except when explicitly stated otherwise.

Closed book. In particular, no notes or books and no calculators. Computers, cell phones, or other electronic devices are only allowed for the purpose of retrieving and submitting the exam or asking me questions, as described in the separate instruction sheet; not for any kind of communication except between you and me. Communication with anyone other than me is prohibited.

1. (a) (10 points) State Morera’s Theorem.
Solution. Theorem 10.17 of Rudin: Let $\Omega \subset \mathbb{C}$ be open, and let $f: \Omega \to \mathbb{C}$ be continuous. Suppose that for every closed triangle in $\Omega$ with boundary path $\gamma$, one has $\int_{\gamma} f(\zeta) \, d\zeta = 0$. Then $f$ is holomorphic on $\Omega$. □

The continuity hypothesis is essential.

(b) (10 points) State Cauchy’s Formula for a convex set.
Solution. Theorem 10.15 of Rudin: Let $\Omega \subset \mathbb{C}$ be a convex open set. Let $\gamma$ be a closed path in $\Omega$, and let $f: \Omega \to \mathbb{C}$ be a holomorphic function. Then the equation
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \text{Ind}_\gamma(z) \cdot f(z)
\]
holds for every $z \in \Omega \setminus \text{Ran}(\gamma)$. □

The factor $\text{Ind}_\gamma(z)$ on the left is essential.

(c) (10 points) State the Fourier Inversion Theorem for functions in $L^2(\mathbb{R})$ (called the Plancherel Theorem in Rudin’s book).
Solution. Theorem 9.13 of Rudin: Let $m$ be Lebesgue measure on $\mathbb{R}$, divided by $\sqrt{2\pi}$. There is a linear map $F: L^2(\mathbb{R}, m) \to L^2(\mathbb{R}, m)$ with the following properties:
(1) If $f \in L^1(\mathbb{R}, m) \cap L^2(\mathbb{R}, m)$, then $F(f) = \hat{f}$.
(2) $\|F(f)\|_2 = \|f\|_2$ for all $f \in L^2(\mathbb{R}, m)$.
(3) $F$ is bijective and preserves the scalar product on $L^2(\mathbb{R}, m)$.
(4) Let $f \in L^2(\mathbb{R}, m)$, and for $a \in [0, \infty)$ define a function $\varphi_a$ by
\[
\varphi_a(t) = \int_{-a}^{a} \exp(-i(x,t))f(x) \, d\mathcal{L}(x)
\]
for $t \in \mathbb{R}$. Then $\varphi_a \in L^2(\mathbb{R}, m)$ for all $a \in [0, \infty)$, and $\lim_{a \to \infty} \|\varphi_a - F(f)\|_2 = 0$.
(5) Let $f \in L^2(\mathbb{R}, m)$, and for $a \in [0, \infty)$ define a function $\psi_a$ by
\[
\psi_a(x) = \int_{-a}^{a} \exp(i(x,t))F(f)(t) \, d\mathcal{L}(t)
\]

Date: 11 June 2020.
for $x \in \mathbb{R}$. Then $\psi_a \in L^2(\mathbb{R}, \mathbb{R})$ for all $a \in [0, \infty)$, and $\lim_{a \to \infty} ||\psi_a - f||_2 = 0$.

2. (30 points) Let $s: \mathbb{C} \to \mathbb{C}$ be the function given by $s(z) = |z|^2$ for $z \in \mathbb{C}$. Determine, with proof, all points $z \in \mathbb{C}$ at which $s$ is differentiable (in the complex sense).

**Solution.** First, we claim that $s'(0) = 0$. To see this, let $\varepsilon > 0$. Set $\delta = \varepsilon$. If $0 < |h| < \delta$, then

$$\left| \frac{s(h) - s(0)}{h} \right| = \frac{|h|^2}{|h|} = |h| < \delta.$$ 

The claim is proved.

Let $z \in \mathbb{C} \setminus \{0\}$. We claim that $s'(z)$ does not exist. Write $z = a + bi$ with $a, b \in \mathbb{R}$. For $t$ running through $\mathbb{R} \setminus \{0\}$, we have

$$\lim_{t \to 0} \frac{s(z + t) - s(z)}{t} = \lim_{t \to 0} \frac{|a + bi + t|^2 - |a + bi|^2}{t} = \lim_{t \to 0} \frac{(a + t)^2 + b^2 - (a^2 + b^2)}{t} = \lim_{t \to 0} \frac{2at + t^2}{t} = 2a,$$

and

$$\lim_{t \to 0} \frac{s(z + it) - s(z)}{it} = \lim_{t \to 0} \frac{|a + (b + t)i|^2 - |a + bi|^2}{it} = \lim_{t \to 0} \frac{a^2 + (b + t)^2 - (a^2 + b^2)}{it} = \lim_{t \to 0} \frac{2bt + t^2}{it} = -2bi.$$ 

Since $a$ and $b$ are real and not both zero, $2a \neq -2bi$. If $s'(z)$ existed, these two limits would have to be equal, so $s'(z)$ does not exist.

Thus $s'(z)$ exists if and only if $z = 0$. $\square$

In the limits in this computation, it is necessary to specify somehow that $t$ is supposed to be real, since, in context, the natural assumption is that $t \in \mathbb{C} \setminus \{0\}$. Unfortunately, standard notation for limits does not make it convenient to include this. (The same issue can arise with $\lim_{t \to \infty} k(t)$, when the question is whether $t$ runs through $\mathbb{Z} \setminus \{0\}$, or, say, $(0, \infty)$. There are problems in which both occur, which people have gotten mostly wrong because the grader could not tell which was intended at which place.)

**Alternate solution for the case $z \neq 0$.** Let $\Omega \subset \mathbb{C}$ be open. The Cauchy-Riemann equations for a function $f: \Omega \to \mathbb{C}$ are as follows. If we set $u(x, y) = \Re(f(x + iy))$ and $v(x, y) = \Im(x + iy)$ for all pairs $(x, y) \in \mathbb{R}^2$ such that $x + iy \in \Omega$, then the Cauchy-Riemann equations are:

$$D_1 u(x, y) = D_2 v(x, y) \quad \text{and} \quad D_2 u(x, y) = -D_1 v(x, y).$$

We proved in class (but this is not in Chapter 10 of Rudin’s book; it is at the beginning of Chapter 11) that if $f$ is complex differentiable at $x + iy$, then (1) holds at $(x, y)$.

For the function at hand, $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$ for all $x, y \in \mathbb{R}$. So (1) becomes

$$2x = 0 \quad \text{and} \quad 0 = -2y.$$
This clearly fails unless \( x = y = 0 \).

The Cauchy-Riemann equations by themselves are not good enough to handle the case \( z = 0 \). To use them to deduce complex differentiability at \( x + iy \), one needs to know that \( f \) is differentiable as a function to \( \mathbb{R}^2 \), that is, that there is an actual linear map \( T_1, T_2 : \mathbb{R}^2 \to \mathbb{R} \) such that, with limits over \((h,k) \in \mathbb{R}^2,\)

\[
\lim_{(h,k) \to 0} \frac{|f(x + h + i(y + k)) - f(x + iy) - [T_1(h,k) + iT_2(h,k)]|}{|h + ik|} = 0.
\]

The following example shows that it is not enough to assume existence of the partial derivatives of \( u \) and \( v \) as above and that (1) holds.

Define \( f : \mathbb{C} \to \mathbb{C} \) by

\[
f(x + iy) = \begin{cases} \frac{xy}{x^2 + y^2} & x + iy \neq 0 \\ 0 & x + iy = 0. \end{cases}
\]

If we set \( u(x,y) = \text{Re}(f(x + iy)) \) and \( v(x,y) = \text{Im}(f(x + iy)) \), then \( D_1u(x,y), D_2u(x,y), D_1v(x,y), \) and \( D_2v(x,y) \) exist for all \( x,y \in \mathbb{R}, \) and (1) holds at 0 (because all the partial derivatives are zero), but \( f \) is not even continuous at 0.

3. (30 points) Let \( h \in C_c(\mathbb{R}) \), the set of continuous complex valued functions on \( \mathbb{R} \) which have compact support. Prove that \( \hat{h} \) is infinitely often differentiable.

**Solution.** Theorem 9.2(f) of Rudin’s book states that if \( f \in L^1(\mathbb{R}) \) and the function \( g(x) = -ixf(x) \) is also in \( L^1(\mathbb{R}) \), then \( \hat{f} \) is differentiable and \( (\hat{f})'(t) = \hat{g}(t) \) for all \( t \in \mathbb{R} \).

We prove by induction that for all \( n \in \mathbb{Z}_{\geq 0} \), the \( n \)-th derivative \( \hat{h}^{(n)} \) exists and is equal to the Fourier transform of the function \( k_n(x) = (-ix)^n h(x) \). The statement for \( n = 0 \) is immediate, since \( h \in C_c(\mathbb{R}) \) implies \( h \in L^1(\mathbb{R}) \), which implies that \( \hat{h} \) is defined, and since \( k_0 = h \). Assuming the statement holds for \( n \), we observe that \( k_n \in C_c(\mathbb{R}) \subseteq L^1(\mathbb{R}) \). Therefore we can apply Theorem 9.2(f) of Rudin’s book with \( f = k_n \) and \( g = k_{n+1} \) see that \( \hat{k}_n \) is differentiable with derivative \( \hat{k}_{n+1} \). Thus, using the induction hypothesis, \( (\hat{h}^{(n)})' \) exists and is equal to \( \hat{k}_{n+1} \).

The proof is written in more detail than is really needed. But just saying “by induction” is not enough; you need to know that \( \hat{h}^{(n)} \) is the Fourier transform of something to which Theorem 9.2(f) of Rudin’s book can be applied.

In fact, \( \hat{f} \) is the restriction to \( \mathbb{R} \) of an entire function. This can be proved by combining Morera’s Theorem and Fubini’s Theorem. The proof is considerably longer, since one must prove as a preliminary step that the function

\[
z \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixz} \, dx
\]

is continuous on \( \mathbb{C} \).

4. (35 points) Let \( f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) be a holomorphic function such that, for \( z \neq 0 \), we have \( |f(z)| \leq 1 + |z|^{-1/2} \). Prove that \( f \) is constant.

**Solution.** Define \( g_0(z) = zf(z) \) for \( z \in \mathbb{C} \setminus \{0\} \). Then \( |g_0(z)| \leq |z| + |z|^{1/2} \) for all \( z \in \mathbb{C} \setminus \{0\} \). Therefore \( g_0 \) is bounded on \( \{z \in \mathbb{C} : |z| < 1\} \). So \( g_0 \) has a removable singularity at 0. That is, there is a holomorphic function \( g : \mathbb{C} \to \mathbb{C} \) such that
The last expression converges to zero as $r \to \infty$. Hence there are $n \in \mathbb{Z}_{>0}$ and a holomorphic function $k \colon \mathbb{C} \to \mathbb{C}$ such that $k(0) \neq 0$ and $g(z) = z^n k(z)$ for all $z \in \mathbb{C}$. Then $f(z) = z^{n-1} k(z)$ for all $z \in \mathbb{C} \setminus \{0\}$. (Take $z^{n-1} = 1$ for all $z$, even $z = 0$, if $n = 1$.) The right hand side is an entire function of $z$. Call it $l$. Then $l$ is bounded on $\{z \in \mathbb{C} : |z| \leq 1\}$ because this set is compact, and on the set $M = \{z \in \mathbb{C} : |z| \geq 1\}$ since $l(z) \leq 1 + |z|^{-1/2} \leq 2$ for $z \in M$. Therefore $l$ is constant by Liouville’s Theorem, and hence $f$ is constant. □

Alternate solution. Define $g_0(z) = f(1/z)$ for $z \in \mathbb{C} \setminus \{0\}$. Then $|g_0(z)| \leq 1 + |z|^{1/2}$ for all $z \in \mathbb{C} \setminus \{0\}$. Therefore $g_0$ is bounded on $\{z \in \mathbb{C} : |z| < 1\}$. So $g_0$ has a removable singularity at $0$. That is, there is a holomorphic function $g \colon \mathbb{C} \to \mathbb{C}$ such that $g(z) = g_0(z)$ for all $z \in \mathbb{C} \setminus \{0\}$. We still have $|g(z)| \leq 1 + |z|^{1/2}$ for all $z \in \mathbb{C}$.

Let $a \in \mathbb{C}$. Let $r > 0$ and apply Cauchy’s Estimates at the first step to get

$$|g'(a)| \leq \frac{1}{r} \sup_{|z| = r} |g(z)| \leq \frac{1}{r} \sup_{|z| = r} (1 + |z|^{1/2}) = \frac{1}{r} \sup_{|z| = r} (1 + |z + a|^{1/2}) \leq \frac{1}{r} \left(1 + (r + |a|)^{1/2}\right).$$

The last expression converges to zero as $r \to \infty$. Therefore $g'(a) = 0$. This holds for all $a \in \mathbb{C}$. Therefore $g$ is constant. It follows that $f$ is constant. □

Second alternate solution. As in the first solution, there is an entire function $g \colon \mathbb{C} \to \mathbb{C}$ such that $g(z) = z f(z)$ for all $z \in \mathbb{C} \setminus \{0\}$. The estimate on $|f(z)|$ implies $|g(z)| \leq |z + |z|^{1/2}$ for all $z \in \mathbb{C} \setminus \{0\}$, hence for all $z \in \mathbb{C}$.

We now apply Cauchy’s Estimates at the point zero to get, for all $r > 0$ and all $n \in \mathbb{Z}_{\geq 0}$,

$$|g^{(n)}(0)| \leq \frac{n!}{r^n} \sup_{|z| = r} |g(z)| \leq \frac{n!}{r^n} \left(1 + r^{1/2}\right).$$

If $n > 1$, the last expression converges to zero as $r \to \infty$, whence $g^{(n)}(0) = 0$. The power series representation therefore implies that there are $a, b \in \mathbb{C}$ such that $g(z) = a + b z$ for all $z \in \mathbb{C}$. The estimate $|g(z)| \leq |z + |z|^{1/2}$ implies $g(0) = 0$, so $a = 0$. Therefore $g(z) = b z$ for all $z \in \mathbb{C}$. So $f(z) = b$ for all $z \in \mathbb{C} \setminus \{0\}$. □

5. (30 points) Suppose $f \colon D \to \mathbb{C}$ is a holomorphic function on the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. If $f$ is injective on $D \setminus \{0\}$, prove that $f$ is injective on $D$.

Solution. Suppose $f$ is not injective on $D$. Then there exists $z_0 \in D \setminus \{0\}$ such that $f(z_0) = f(0)$. Choose disjoint open subsets $V, W \subset D$ such that $0 \in V$ and $z_0 \in W$. Clearly $f$ is not constant, so $f$ is an open mapping, and $f(V)$ and $f(W)$ are open.

Now $f(0) \in f(V) \cap f(W)$, and $\{f(0)\}$ is not open, so there is some $w \in f(V) \cap f(W)$ with $w \neq f(0)$. There are $a \in V$ and $b \in W$ such that $f(a) = f(b) = w$. Clearly $a \neq b$ and neither $a$ nor $b$ is zero. So $f$ is not injective on $D \setminus \{0\}$. □

Alternate solution. If $f'$ is constant, then either $f$ is constant or there are $a, b \in \mathbb{C}$ such that $f(z) = a z + b$ for all $z \in D$. In either case, the statement of the problem is clear. So we can assume $f'$ is not constant. Replacing $f$ by $f - f(0)$, we can further assume that $f(0) = 0$.

Suppose $f$ is not injective on $D$. Then there exists $z_0 \in D \setminus \{0\}$ such that $f(z_0) = 0$. Choose $r$ such that $|z_0| < r < 1$. Set $C = \frac{1}{2} \inf_{|z|=r} |f(z)|$. Since $f$ is
injective on $D \setminus \{0\}$ and $f(z_0) = 0$, it follows that $f(z) \neq 0$ when $|z| = r$, so $C > 0$. Choose $\delta > 0$ such that $\delta < r$ and such that $|z| < \delta$ implies $|f(z)| < C$. Since $f'$ is not constant, its zeros are isolated, and there is $w \in D$ such that $0 < |w| < \delta$ and $f'(w) \neq 0$. Therefore the function $g(z) = f(z) - f(w)$ has a simple zero at $w$. Since $w \neq 0$, the injectivity hypothesis implies that $g$ has no other zeros in $D$.

When $|z| = r$, we have

$$|g(z) - f(z)| = |f(w)| < C < |f(z)|.$$  

So Rouché’s Theorem implies that $f$ and $g$ have the same number of zeros in $\{z \in \mathbb{C} : |z| < r\}$, counting multiplicity. But we saw that $g$ has only one zero, which has multiplicity 1, while $f$ has at least two zeros. This contradiction shows that $f$ is injective on $D$. □

Second alternate solution (sketch). In the alternate solution, instead of using $f'$ not constant to choose $w$, choose $w$ arbitrarily and use the fact that if $f - f(w)$ has a zero at $w$ of multiplicity more than 1, then $f$ is not injective on any neighborhood of $w$. □

6. (45 points) Let $f$ be an entire function. Suppose that there are constants $C$ and $M$ such that $|f(z)| \leq C + M|z|$ whenever $\text{Im}(z) \geq 0$, and further suppose that $\lim_{r \to \infty} f(rz)$ exists whenever $\text{Im}(z) > 0$. Prove that

$$\lim_{r \to \infty} \int_{-r}^{r} \frac{f(x)}{1 + x^2} \, dx$$

exists.

Solution. For $r > 0$, define paths $\gamma_r : [-r,r] \to \mathbb{C}$, $\rho_r : [0, \pi] \to \mathbb{C}$ and $\sigma_r : [\pi, 2\pi] \to \mathbb{C}$ by $\gamma_r(t) = t$ for $t \in [-r,r]$, $\rho_r(t) = e^{it}$ for $t \in [0,\pi]$, and $\sigma_r(t) = e^{it}$ for $t \in [\pi,2\pi]$. Then $[\gamma_r] + [\rho_r] + [\sigma_r]$ and $[\gamma_r] - [\sigma_r]$ are cycles.

The function

$$g(z) = \frac{f(z)}{1 + z^2}$$

is meromorphic on $\mathbb{C}$, with (possibly removable) singularities at $i$ and $-i$. We have $\text{Ind}_{[\gamma_r] + [\rho_r]}(-i) = 0$, because the lower half plane is an unbounded set which contains $-i$ and is disjoint from $\text{Ran}([\gamma_r] + [\rho_r])$. Similarly $\text{Ind}_{[\gamma_r] - [\sigma_r]}(i) = 0$. For $r > 1$, Theorem 10.11 of Rudin implies that $\text{Ind}_{[\rho_r] + [\sigma_r]}(i) = 1$, so

$$\text{Ind}_{[\gamma_r] + [\rho_r]}(i) = \text{Ind}_{[\gamma_r] - [\sigma_r]}(i) + \text{Ind}_{[\rho_r] + [\sigma_r]}(i) = 1.$$  

It follows from the Residue Theorem that the function

$$r \mapsto \int_{[\gamma_r] + [\rho_r]} \frac{f(z)}{1 + z^2} \, dz$$

is constant on $(1, \infty)$ (with value $2\pi i \text{Res}(g;i)$). Therefore it suffices to prove that

$$\lim_{r \to \infty} \int_{\rho_r} \frac{f(z)}{1 + z^2} \, dz$$

exists.

We have

$$\int_{\rho_r} \frac{f(z)}{1 + z^2} \, dz = \int_{0}^{\pi} \frac{f(re^{it})ire^{it}}{1 + (re^{it})^2} \, dt.$$
Set
\[ h_r(t) = \frac{f(re^{it})ire^{it}}{1+(re^{it})^2} \]
for \( t \in [0,\pi] \) and \( r \in (1,\infty) \). For \( t \in (0,\pi) \), we have \( \text{Im}(e^{it}) > 0 \). Therefore \( \lim_{r \to \infty} f(re^{it}) \) exists, and it follows that \( \lim_{r \to \infty} h_r(t) = 0 \). Thus \( h_r(t) \to 0 \) pointwise almost everywhere on \([0,\pi]\). Also, for \( r > 2 \) we have
\[ |1 + (re^{it})^2| \geq r^2 - 1 > \frac{1}{2}r^2, \]
so
\[ |h_r(t)| < \frac{2|f(re^{it})| \cdot r}{r^2} \leq \frac{2(C + Mr)}{r} \leq C + 2M. \]
Since the constant function \( t \mapsto C + 2M \) is integrable on \([0,2\pi]\), we can apply the Dominated Convergence Theorem to conclude that, for every sequence \((r_n)_{n \in \mathbb{Z}_+}\) in \((2,\infty)\) with \( \lim_{n \to \infty} r_n = 0 \), we have
\[ \lim_{n \to \infty} \int_{\rho_{r_n}} \frac{f(z)}{1+z^2} \, dz = 0. \]
Therefore
\[ \lim_{r \to \infty} \int_{\rho_r} \frac{f(z)}{1+z^2} \, dz = 0. \]
This completes the proof. \( \square \)

One must use sequences in the Dominated Convergence Theorem, since the Dominated Convergence Theorem does not work for more general nets.

One can use what in lecture was called the “path changing lemma” (Theorem 10.37 of Rudin’s book) to prove that \( \text{Ind}_{\gamma_r + \partial D}(i) = 1 \), but the method described is simpler.

(There really are functions \( f \) satisfying the hypotheses, such as \( f(z) = 1 \), \( f(z) = e^{iz} \), and \( f(z) = ze^{-z} \). The function \( f(z) = z \) does not satisfy the hypotheses.)

Extra Credit. (40 extra credit points.) Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Prove that the series
\[ \sum_{n=1}^{\infty} \frac{z^{3n+1}}{2^n} \]
converges to a continuous function \( f(z) \) on \( \overline{D} \) which is holomorphic on \( D \). Further prove (almost all the credit is for this part) that there does not exist any pair \((\Omega, g)\) in which \( \Omega \) is a region with \( \Omega \cap \overline{D} \neq \emptyset \) and \( g \) is a holomorphic function on \( \Omega \) such that \( g|_{\Omega \cap D} = f|_{\Omega \cap D} \). (Grading will be considerably stricter than on the regular problems.)

Solution. The series converges uniformly on \( \overline{D} \) because \( |z^{3n+1}/2^n| \leq \frac{1}{2^n} \) for \( z \in \overline{D} \) and \( \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \). Therefore \( f \) is continuous.

If we write \( f(z) = \sum_{n=0}^{\infty} c_n z^n \), then \( |c_n| \leq 1 \) for all \( n \). It is immediate that the series has radius of convergence at least 1. (This also follows from the previous paragraph.) Therefore \( f \) is holomorphic on \( D \).

We claim that \( \lim_{r \to 1^-} \text{Re}(f'(rz)) = \infty \) for every \( z \) of the form \( \exp(2\pi ik/3l) \) with \( k \in \mathbb{Z} \) and \( l \in \mathbb{Z}_{\geq 0} \). (This is the main step in proving the last statement in
the problem.) By the theorem on term by term differentiation of power series, we have
\[ f'(z) = \sum_{n=1}^{\infty} \frac{(3^n + 1)z^n}{2^n} \]
for all \( z \in D \). Fix \( k \in \mathbb{Z} \) and \( l \in \mathbb{Z}_{\geq 0} \), and set \( z = \exp(2\pi ik/3^l) \). Suppose \( 0 < r < 1 \).
Then for \( n \geq l \),
\[ \frac{(3^n + 1)(rz)^3}{2^n} = \frac{(3^n + 1)r^3}{2^n} \]
Set
\[ M_0 = \sum_{n=1}^{l} \frac{3^n + 1}{2^n} \]
For any \( M \in \mathbb{R} \) there are \( r_0 < 1 \) and \( n \geq l \) such that
\[ \frac{(3^n + 1)r_{0}^3}{2^n} > M + M_0, \]
and for \( r_0 < r < 1 \) we have
\[ \text{Re}(f'(rz)) \geq \frac{(3^n + 1)r_{0}^3}{2^n} - \text{Re} \left( \sum_{n=1}^{l} \frac{(3^n + 1)(rz)^3}{2^n} \right) \]
\[ \geq \frac{(3^n + 1)r_{0}^3}{2^n} - \sum_{n=1}^{l} \frac{(3^n + 1)|rz|^3}{2^n} > (M + M_0) - M_0 = M. \]
This completes the proof that \( \lim_{r \to 1^-} \text{Re}(f'(rz)) = \infty \).
Now suppose \( \Omega \) is a region with \( \Omega \cap \partial D \neq \emptyset \), and suppose \( g \) is a holomorphic function on \( \Omega \) such that \( g|_{\Omega \cap D} = f|_{\Omega \cap D} \). Then we can choose \( k \in \mathbb{Z} \) and \( l \in \mathbb{Z}_{\geq 0} \) such that \( z = \exp(2\pi ik/3^l) \in \Omega \). It follows from the previous paragraph that \( g' \) is not bounded on any neighborhood of \( z \). This contradicts continuity of \( g' \) at \( z \). \( \square \)

**Remark:** For the last part, it is *not* enough to show that the radius of convergence of the power series is at most 1, and very little credit will be given for only doing this. For example, the series \( \sum_{n=1}^{\infty} z^n \) has radius of convergence 1, but one can take \( \Omega = \mathbb{C} \setminus \{1\} \) and \( g(z) = (1 - z)^{-1} \).