This assignment is due Monday 13 April at 10:00 pm.

Conventions on measures: m is ordinary Lebesgue measure, \( m = (2\pi)^{-1/2}m \), and in expressions of the form \( \int f(x) dx \), ordinary Lebesgue measure is assumed.

**Problem 1.** This problem is about giving direct proofs of results on approximate identities.

1. Let \( X \) be a locally compact \( \sigma \)-compact Hausdorff space. Prove that there is a sequence \( (g_n)_{n \in \mathbb{Z}^+} \) in \( C^0(X) \) consisting of functions with compact support and with values in \([0, 1]\) such that for every \( f \in C^0(X) \) we have \( \lim_{n \to \infty} \| g_n f - f \|_{\infty} = 0 \).

2. Let \( g \in L^1(\mathbb{R}) \) satisfy \( g(x) = 0 \) for all \( x \in \mathbb{R} \setminus [-1, 1] \), \( g(x) \geq 0 \) for all \( x \in \mathbb{R} \), and \( \int \! g \, d\mathcal{M} = 1 \). For \( n \in \mathbb{Z}^+ \) and \( x \in \mathbb{R} \), define \( g_n(x) = ng(nx) \). Let \( p \in [1, \infty) \). Prove that for every \( f \in L^p(\mathbb{R}) \) we have \( \lim_{n \to \infty} \| g_n * f - f \|_p = 0 \).

In part (1), if \( X \) is not \( \sigma \)-compact, one needs a net instead of a sequence. You will need to prove that there are compact subsets \( K_1, K_2, \ldots \subset X \) such that

\[
K_1 \subset \text{int}(K_2) \subset K_2 \subset \text{int}(K_3) \subset K_3 \subset \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} K_n = X.
\]

(I couldn’t find this explicitly in Rudin’s book, but maybe I didn’t look in the right place.)

In part (2), I suggest first proving the result for \( f \in C_c(\mathbb{R}) \). You will also need the following result, which you may use without proof.

**Proposition 2.** Let \( p \in [1, \infty] \), let \( f \in L^1(\mathbb{R}) \), and let \( g \in L^p(\mathbb{R}) \). Then \( f * g \in L^p(\mathbb{R}) \), and \( \| f * g \|_p \leq \| f \|_1 \| g \|_p \).

**Problem 3.** Give a “direct” proof of the following part of Theorem 9.6 of Rudin’s book: if \( f \in L^1 \) then \( \hat{f} \in C_0(\mathbb{R}) \). That is, prove this first when \( f \) is the characteristic function of a bounded interval, use this result and approximation to prove \( \hat{f} \in C_0(\mathbb{R}) \) when \( f \in C_c(\mathbb{R}) \), and then use approximation to prove \( \hat{f} \in C_0(\mathbb{R}) \) for general \( f \in L^1 \).

You will need \( \| \hat{f} \|_{\infty} \leq \| f \|_1 \). This proof takes longer than Rudin’s proof, but the methods are useful much more generally, and the first step explains why the result is even true.

Problem 4 counts as two ordinary problems.

**Problem 4** (Rudin, Chapter 9, Problem 13abc). For \( c \in (0, \infty) \) define \( f_c : \mathbb{R} \to \mathbb{C} \) by \( f_c(x) = \exp(-cx^2) \) for \( x \in \mathbb{R} \).

1. Compute \( f_c \).
(2) Prove that there exists a unique \( c \in (0, \infty) \) such that \( \hat{f}_c = f_c \).

(3) Let \( a, b \in (0, \infty) \). Prove that there exist \( \gamma \) and \( c \) such that \( f_a * f_b = \gamma f_c \), and find explicit formulas for \( \gamma \) and \( c \) in terms of \( a \) and \( b \).

Hint for part (1): Let \( g = \hat{f}_c \). Then an integration by parts gives \( 2cg'(t) + tg(t) = 0 \) for all \( t \). If you use this method, you will need to prove (directly, or by citing theorems) that this equation, together with other information you have, determines \( g \) uniquely.

You may take as known the result that \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \). (This is proved by writing the square of the integral as \( \int_{\mathbb{R}^2} e^{-x^2 - y^2} \, dx \, dy \) and computing it with polar coordinates in \( \mathbb{R}^2 \).)