Little proofreading has been done.
Some parts of problems have several different solutions.

1. Problem 1: Rudin, Chapter 10, Problem 2

Problem 1.1. Let $f$ be an entire function. Suppose that for every $a \in \mathbb{C}$, in the power series representation

$$f(z) = \sum_{n=0}^{\infty} c_{n,a}(z-a)^n,$$

there is $n \in \mathbb{Z}_{\geq 0}$ such that $c_{n,a} = 0$. Prove that $f$ is a polynomial.

Hint: $n!c_{n,a} = f^{(n)}(a)$.

This is Problem 2 in Chapter 10 of Rudin’s book. Rudin wrote (1) as “$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$”. Suppressing the dependence on $a$ in the notation for the coefficients makes proper writing of both the problem and its solution awkward.

Solution. For $n \in \mathbb{Z}_{\geq 0}$, set

$$Z_n = \{ a \in \mathbb{C} : c_{n,a} = 0 \}.$$

By hypothesis, we have $\bigcup_{n=0}^{\infty} Z_n = \mathbb{C}$. Therefore there exists $n \in \mathbb{Z}_{\geq 0}$ such that $Z_n$ is uncountable. Since $Z_n = \bigcup_{m=1}^{\infty} (B_m(0) \cap Z_n)$, it follows that there is $m$ such that $B_m(0) \cap Z_n$ is infinite. Therefore $B_m(0) \cap Z_n$ has a cluster point in $\mathbb{C}$. So $Z_n$ has a cluster point in $\mathbb{C}$. Since $f^{(n)}(z) = 0$ for all $z \in Z_n$, it follows that $f^{(n)} = 0$. Therefore $f$ is a polynomial (of degree at most $n-1$). \qed

2. Problem 2: Rudin, Chapter 10, Problem 3

Problem 2.1. Suppose that $f$ and $g$ are entire functions, and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. What conclusion can you draw?

Solution. The conclusion is that there is a constant $c$ such that $|c| \leq 1$ and $f = cg$.

No stronger conclusion is possible, since, if $g$ is an entire function and $c \in \mathbb{C}$ satisfies $|c| \leq 1$, then $f = cg$ is an entire function such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$.

We prove the claimed conclusion. Set $Z = \{ z \in \mathbb{C} : g(z) = 0 \}$. There are two cases. First, assume that $Z$ has a limit point in $\mathbb{C}$. Then $g = 0$. Therefore also $f = 0$, and $c = 1$ (or $c = 0$, or $c = \frac{1}{n}$) will do.

Suppose now that $Z$ has no limit point in $\mathbb{C}$. Define a holomorphic function $h_0$ on $\mathbb{C} \setminus Z$ by $h_0(z) = f(z)/g(z)$ for all $z \in \mathbb{C} \setminus Z$. Then $|h_0(z)| \leq 1$ for all $z \in \mathbb{C} \setminus Z$. Since the points of $Z$ are isolated, it follows that they are all removable singularities of $h_0$. Therefore there exists an entire function $h$ such that $h|_{\mathbb{C} \setminus Z} = h_0$. By continuity, $|h(z)| \leq 1$ for all $z \in \mathbb{C}$. Therefore Liouville’s Theorem implies that

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there is a constant $c \in \mathbb{C}$ such that $h(z) = c$ for all $z \in \mathbb{C}$. Clearly $|c| \leq 1$. It follows that $f(z) = cg(z)$ for all $z \in \mathbb{C} \setminus Z$. By continuity, we must have $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

\[\square\]

3. Problem 3: Rudin, Chapter 10, Problem 4

**Problem 3.1.** Let $f$ be an entire function. Suppose that there are constants $A, B > 0$ and $k \in \mathbb{Z}_{>0}$ such that $|f(z)| \leq A + B|z|^k$ for all $z \in \mathbb{C}$. Prove that $f$ is a polynomial.

We give three solutions. The first is probably the intended solution, but both the others have been used.

**Solution.** Since $f$ is entire, there are $c_0, c_1, \ldots \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$.

Let $n \in \mathbb{Z}_{>0}$ satisfy $n > k$. For all $r > 0$ we have

$$\sup_{z \in B_r(0)} |f(z)| \leq A + Br^k.$$ 

Combining this estimate with Cauchy’s Estimates in the second step, we get

$$|n!c_n| = |f^{(n)}(0)| \leq \frac{n!(A + Br^k)}{r^n}.$$ 

Since

$$\lim_{r \to \infty} \frac{n!(A + Br^k)}{r^n} = 0,$$

it follows that $c_n = 0$.

Therefore $f(z) = \sum_{n=0}^{k} c_n z^n$ for all $z \in \mathbb{C}$. That is, $f$ is a polynomial (of degree at most $k$). \[\square\]

**Second solution.** Let $a \in \mathbb{C}$. Then for $r > |a|$ and $z \in B_r(a)$, we have

$$|f(z)| \leq A + B|z|^k \leq A + B(2r)^k = A + 2^k Br^k.$$ 

Combining this estimate with Cauchy’s Estimates, we get

$$|f^{(k+1)}(a)| \leq \frac{(k+1)!(A + 2^k Br^k)}{r^{k+1}}.$$ 

Since

$$\lim_{r \to \infty} \frac{(k+1)!(A + 2^k Br^k)}{r^{k+1}} = 0,$$

it follows that $f^{(k+1)}(a) = 0$.

Since $a \in \mathbb{C}$ is arbitrary, we have shown that $f^{(k+1)} = 0$. Therefore $f^{(n)} = 0$ for all $n > k$. Since $f$ is entire, there are $c_0, c_1, \ldots \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$. We have $c_n = f^{(n)}(0)/n!$ for all $n > k$, so $f(z) = \sum_{n=0}^{k} c_n z^n$ for all $z \in \mathbb{C}$. That is, $f$ is a polynomial (of degree at most $k$). \[\square\]

**Third solution.** We prove the result by induction on $k \in \mathbb{Z}_{\geq 0}$. If $k = 0$, then $f$ is bounded, so $f$ is constant by Liouville’s Theorem.

Suppose now that the result is known for some $k \in \mathbb{Z}_{\geq 0}$, and that $f$ is an entire function and there are constants $A, B > 0$ such that $|f(z)| \leq A + B|z|^{k+1}$ for all $z \in \mathbb{C}$. For $z \in \mathbb{C} \setminus \{0\}$, define

$$g_0(z) = \frac{f(z) - f(0)}{z}.$$
Then $g$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Also $\lim_{z \to 0} g(z) = f'(0)$. In particular, the limit exists, so $g$ is bounded on $B_1(0) \setminus \{0\}$, and must therefore have a removable singularity at $0$. Thus there is an entire function $g$ such that $g|_{\mathbb{C}\setminus\{0\}} = g_0$.

Define

$$M = \sup_{z \in \overline{B_1(0)}} |g(z)|,$$

which is finite because $\overline{B_1(0)}$ is compact. Set $A_0 = \max(M, |f(0)| + A)$. Let $z \in \mathbb{C}$. If $|z| < 1$, then

$$|g(z)| \leq M \leq A_0 + B|z|^k.$$

If $|z| \geq 1$, then

$$|g(z)| \leq \frac{|f(z)|}{|z|} + \frac{|f(0)|}{|z|} \leq \frac{A}{|z|} + B|z|^k + \frac{|f(0)|}{|z|} \leq A + B|z|^k + |f(0)| \leq A_0 + B|z|^k.$$

Therefore $g$ satisfies the induction hypothesis. So $g$ is a polynomial. Hence $f(z) = zg(z) + f(0)$ is also a polynomial.

4. Problem 4: Rudin, Chapter 10, Problem 6

**Problem 4.1.** Prove that there is a region $\Omega$ such that $\exp(\Omega) = B_1(1)$. Prove that there are are many such choices of $\Omega$. Prove that, for any such choice of $\Omega$, the restriction $\exp|_{\Omega}$ is injective. Fix one such choice of $\Omega$, and define $\log: B_1(1) \to \Omega$ to be the inverse function of $\exp|_{\Omega}$. Prove that $\log'(z) = z^{-1}$. Find the coefficients $a_n$ in the expansion

$$\frac{1}{z} = \sum_{n=0}^{\infty} a_n(z-1)^n,$$

and hence find the coefficients $c_n$ in the expansion

$$\log(z) = \sum_{n=0}^{\infty} c_n(z-1)^n.$$

In which other disks can this be done?

**Solution.** The following arguments are valid for any open ball $B_r(a)$ which does not contain $0$.

We will use the standard facts about $\exp$ and $\log$ in calculus of a real variable, and the polar form $z = r \exp(i\theta)$ of a complex number $z$, with unique $r \geq 0$ and, if $r > 0$, uniquely determined $\theta \mod 2\pi \mathbb{Z}$. We will also use the following facts about $\exp$:

1. $\exp(z_1 + z_2) = \exp(z_1) \exp(z_2)$ for every $z_1, z_2 \in \mathbb{C}$. (This can be gotten from the power series.)
2. $\exp$ is periodic with period $2\pi i$. (This is well known.)
3. For every $z \in \mathbb{C} \setminus \{0\}$, there is $b \in \mathbb{C}$ such that $\exp(b) = z$. (Write $z = r \exp(i\theta)$ with $r \in (0, \infty)$ and $\theta \in \mathbb{R}$. Then take $b = \log(r) + i\theta$, using the usual definition of $\log$: $(0, \infty) \to \mathbb{R}$.)

We first claim that if $\Omega$ is a connected set such that $\exp(\Omega) \subset B_r(a)$, then $\exp|_{\Omega}$ is injective. Let $T = \{-ta: t \in [0, \infty)\}$. Write $a = r \exp(i\theta)$ with $r \in (0, \infty)$ and $\theta \in \mathbb{R}$. Then $\exp(z) \in T$ if and only if $\text{Im}(z) \in \theta + 2\pi \mathbb{Z}$. (This is easily checked using the facts above.) The connected components of

$$\{z \in \mathbb{C}: \text{Im}(z) \in \theta + 2\pi \mathbb{Z}\}$$

are

$$\{z \in \mathbb{C}: \text{Im}(z) \in \theta + 2\pi \mathbb{Z}\}.$$
are the strips
\[ S_n = \{ z \in \mathbb{C} : \theta + 2\pi n < \operatorname{Im}(z) < \theta + 2\pi(n + 1) \} \]
for \( n \in \mathbb{Z} \). Since \( T \cap B_r(a) = \emptyset \), there is \( n \in \mathbb{Z} \) such that \( \Omega \subset S_n \). One easily checks (again using the facts above) that \( \exp|_{S_n} \) is injective. In particular, \( \exp|_{\Omega} \) is injective.

It also follows that if \( \Omega \subset \mathbb{C} \) is one region such that \( \exp(\Omega) = B_r(a) \), then the collection of all such regions is
\[ \{ 2\pi in + \Omega : n \in \mathbb{Z} \} . \]

Now we prove the existence of \( \Omega \) and the formulas involving the function \( \log \).

Choose any \( b \in \mathbb{C} \) such that \( \exp(b) = a \). Theorem 10.14 of Rudin provides a holomorphic function \( g_0 : B_r(a) \to \mathbb{C} \) such that \( g_0'(z) = z^{-1} \) for all \( z \in B_r(a) \). Set \( g(z) = g_0(z) - g_0(a) + b \). Then also \( g'(z) = z^{-1} \) for all \( z \in B_r(a) \). Furthermore \( g(a) = b \).

Define a holomorphic function \( h : B_r(a) \to \mathbb{C} \) by
\[ h(z) = \frac{\exp(g(z))}{z} . \]
(Note that \( 0 \not\in B_r(a) \).) Using \( g'(z) = z^{-1} \), we get
\[ h'(z) = \frac{\exp(g(z))g'(z)z - \exp(g(z))}{z} = 0 \]
for all \( z \in B_r(a) \). It follows that \( h \) is constant. (Consider the form the power series for \( h \) must take.) Check that \( h(a) = 1 \). It follows that \( \exp(g(z)) = z \) for all \( z \in B_r(a) \). We can now take \( \Omega = g(B_r(a)) \), which is open by the Open Mapping Theorem and connected because \( g \) is continuous and \( B_r(a) \) is connected. This proves the existence of \( \Omega \). Moreover, \( g : B_r(a) \to \Omega \) is surjective by definition and injective because \( \exp(g(z)) = z \) for all \( z \in B_r(a) \). Therefore \( g \) is the inverse of \( \exp|_{\Omega} \), which shows that the derivative of the inverse of \( \exp|_{\Omega} \) is \( z^{-1} \).

It remains only to compute the series in the case \( a = 1 \). Whenever \( w \in \mathbb{C} \) with \( |w| < 1 \), we have
\[ \sum_{n=0}^{\infty} w^n = \frac{1}{1 - w} . \]
Putting \( w = 1 - z \), we get, for \( |z - 1| < 1 \),
\[ \frac{1}{z} = \sum_{n=0}^{\infty} (1 - z)^n = \sum_{n=0}^{\infty} (-1)^n (z - 1)^n . \]
The series
\[ \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} (z - 1)^n \]
has radius of convergence 1 and its term by term derivative is the series for \( z^{-1} \). Therefore it differs from \( g(z) \) be a constant. Taking the number \( b \) with \( \exp(b) = 1 \) to be \( b = 0 \), we find that the constant is zero by comparing the values at 0. Therefore
\[ \log(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} (z - 1)^n \]
when \( |z - 1| < 1 \). □
The proof given above is not what most students have done. Instead, most people have proceeded as follows. One uses the formula $\exp(x + iy) = e^x (\cos(y) + i \sin(y))$ to find a region $\Omega$ such that $\exp$ defines a bijection from $\Omega$ to $B_1(1)$. In terms of the standard real version of the logarithm function (also denoted $\log$), the explicit description of $\Omega$ (not really needed to solve the problem) is

$$\Omega = \{ x + iy : x, y \in \mathbb{R}, -\frac{\pi}{2} < y < \frac{\pi}{2}, \text{ and } x < \log(2 \cos(y)) \}.$$  

One then uses the Inverse Function Theorem to show that $\log = (\exp|_{\Omega})^{-1}$ is holomorphic, and one differentiates the equation $\exp(\log(z)) = z$ to determine its derivative.