MATH 618 (SPRING 2020, PHILLIPS): SOLUTIONS TO HOMEWORK 6

Little proofreading has been done.

Some parts of problems have several different solutions.

1. Problem 1: Rudin, Chapter 10, Problem 2

Problem 1.1. Let f be an entire function. Suppose that for every $a \in \mathbb{C}$, in the power series representation

$$f(z) = \sum_{n=0}^{\infty} c_{n,a} (z-a)^n, \tag{1}$$

there is $n \in \mathbb{Z}_{\geq 0}$ such that $c_{n,a} = 0$. Prove that f is a polynomial.

Hint: $n! c_{n,a} = f^{(n)}(a)$.

This is Problem 2 in Chapter 10 of Rudin's book. Rudin wrote (1) as "$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$". Suppressing the dependence on a in the notation for the coefficients makes proper writing of both the problem and its solution awkward.

Solution. For $n \in \mathbb{Z}_{\geq 0}$, set

$$Z_n = \big\{ a \in \mathbb{C} \colon c_{n,a} = 0 \big\}.$$

By hypothesis, we have $\bigcup_{n=0}^{\infty} Z_n = \mathbb{C}$. Therefore there exists $n \in \mathbb{Z}_{\geq 0}$ such that Z_n is uncountable. Since $Z_n = \bigcup_{m=1}^{\infty} (B_m(0) \cap Z_n)$, it follows that there is m such that $B_m(0) \cap Z_n$ is infinite. Therefore $B_m(0) \cap Z_n$ has a cluster point in $\mathbb{C}$. So Z_n has a cluster point in $\mathbb{C}$. Since $f^{(n)}(z) = 0$ for all $z \in Z_n$, it follows that $f^{(n)} = 0$. Therefore f is a polynomial (of degree at most $n-1$). ☐

2. Problem 2: Rudin, Chapter 10, Problem 3

Problem 2.1. Suppose that f and g are entire functions, and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. What conclusion can you draw?

Solution. The conclusion is that there is a constant c such that $|c| \leq 1$ and $f = cg$.

No stronger conclusion is possible, since, if g is an entire function and $c \in \mathbb{C}$ satisfies $|c| \leq 1$, then $f = cg$ is an entire function such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$.

We prove the claimed conclusion. Set $Z = \big\{ z \in \mathbb{C} \colon g(z) = 0 \big\}$. There are two cases. First, assume that Z has a limit point in $\mathbb{C}$. Then $g = 0$. Therefore also $f = 0$, and $c = 1$ (or $c = 0$, or $c = \frac{1}{\pi}$) will do.

Suppose now that Z has no limit point in $\mathbb{C}$. Define a holomorphic function h_0 on $\mathbb{C} \setminus Z$ by $h_0(z) = f(z)/g(z)$ for all $z \in \mathbb{C} \setminus Z$. Then $|h_0(z)| \leq 1$ for all $z \in \mathbb{C} \setminus Z$. Since the points of Z are isolated, it follows that they are all removable singularities of h_0. Therefore there exists an entire function h such that $h|_{\mathbb{C} \setminus Z} = h_0$. By continuity, $|h(z)| \leq 1$ for all $z \in \mathbb{C}$. Therefore Liouville's Theorem implies that

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there is a constant $c \in \mathbb{C}$ such that $h(z) = c$ for all $z \in \mathbb{C}$. Clearly $|c| \leq 1$. It follows that $f(z) = cg(z)$ for all $z \in \mathbb{C} \setminus Z$. By continuity, we must have $f(z) = cg(z)$ for all $z \in \mathbb{C}$. $\square$

3. Problem 3: Rudin, Chapter 10, Problem 4

Problem 3.1. Let f be an entire function. Suppose that there are constants $A, B > 0$ and $k \in \mathbb{Z}_{>0}$ such that $|f(z)| \leq A + B|z|^k$ for all $z \in \mathbb{C}$. Prove that f is a polynomial.

We give three solutions. The first is probably the intended solution, but both the others have been used.

Solution. Since f is entire, there are $c_0, c_1, \ldots \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$.

Let $n \in \mathbb{Z}_{>0}$ satisfy $n > k$. For all $r > 0$ we have

$$\sup_{z \in B_r(0)} |f(z)| \leq A + Br^k.$$

Combining this estimate with Cauchy's Estimates in the second step, we get

$$|n! c_n| = |f^{(n)}(0)| \leq \frac{n!(A + Br^k)}{r^n}.$$

Since

$$\lim_{r \to \infty} \frac{n!(A + Br^k)}{r^n} = 0,$$

it follows that $c_n = 0$.

Therefore $f(z) = \sum_{n=0}^{k} c_n z^n$ for all $z \in \mathbb{C}$. That is, f is a polynomial (of degree at most k). $\square$

Second solution. Let $a \in \mathbb{C}$. Then for $r > |a|$ and $z \in B_r(a)$, we have

$$|f(z)| \leq A + B|z|^k \leq A + B(2r)^k = A + 2^k Br^k.$$

Combining this estimate with Cauchy's Estimates, we get

$$|f^{(k+1)}(a)| \leq \frac{(k+1)!(A + 2^k Br^k)}{r^{k+1}}.$$

Since

$$\lim_{r \to \infty} \frac{(k+1)!(A + 2^k Br^k)}{r^{k+1}} = 0,$$

it follows that $f^{(k+1)}(a) = 0$.

Since $a \in \mathbb{C}$ is arbitrary, we have shown that $f^{(k+1)} = 0$. Therefore $f^{(n)} = 0$ for all $n > k$. Since f is entire, there are $c_0, c_1, \ldots \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$. We have $c_n = f^{(n)}(0)/n!$ for all $n > k$, so $f(z) = \sum_{n=0}^{k} c_n z^n$ for all $z \in \mathbb{C}$. That is, f is a polynomial (of degree at most k). $\square$

Third solution. We prove the result by induction on $k \in \mathbb{Z}_{\geq 0}$. If $k = 0$, then f is bounded, so f is constant by Liouville's Theorem.

Suppose now that the result is known for some $k \in \mathbb{Z}_{\geq 0}$, and that f is an entire function and there are constants $A, B > 0$ such that $|f(z)| \leq A + B|z|^{k+1}$ for all $z \in \mathbb{C}$. For $z \in \mathbb{C} \setminus \{0\}$, define

$$g_0(z) = \frac{f(z) - f(0)}{z}.$$

Then g is holomorphic on $\mathbb{C} \setminus \{0\}$. Also $\lim_{z\to 0} g(z) = f'(0)$. In particular, the limit exists, so g is bounded on $B_1(0) \setminus \{0\}$, and must therefore have a removable singularity at 0. Thus there is an entire function g such that $g|_{\mathbb{C}\setminus\{0\}} = g_0$.

Define

$$M = \sup_{z \in \overline{B_1(0)}} |g(z)|,$$

which is finite because $\overline{B_1(0)}$ is compact. Set $A_0 = \max(M, |f(0)| + A)$. Let $z \in \mathbb{C}$. If $|z| < 1$, then

$$|g(z)| \leq M \leq A_0 + B|z|^k.$$

If $|z| \geq 1$, then

$$|g(z)| \leq \frac{|f(z)|}{|z|} + \frac{|f(0)|}{|z|} \leq \frac{A}{|z|} + B|z^k| + \frac{|f(0)|}{|z|} \leq A + B|z^k| + |f(0)| \leq A_0 + B|z|^k.$$

Therefore g satisfies the induction hypothesis. So g is a polynomial. Hence $f(z) = zg(z) + f(0)$ is also a polynomial. □

4. Problem 4: Rudin, Chapter 10, Problem 6

Problem 4.1. Prove that there is a region Ω such that $\exp(\Omega) = B_1(1)$. Prove that there are are many such choices of Ω. Prove that, for any such choice of Ω, the restriction $\exp|_\Omega$ is injective. Fix one such choice of Ω, and define $\log\colon B_1(1) \to \Omega$ to be the inverse function of $\exp|_\Omega$. Prove that $\log'(z) = z^{-1}$. Find the coefficients a_n in the expansion

$$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-1)^n,$$

and hence find the coefficients c_n in the expansion

$$\log(z) = \sum_{n=0}^{\infty} c_n (z-1)^n.$$

In which other disks can this be done?

Solution. The following arguments are valid for any open ball $B_r(a)$ which does not contain 0.

We will use the standard facts about exp and log in calculus of a real variable, and the polar form $z = r\exp(i\theta)$ of a complex number z, with unique $r \geq 0$ and, if $r > 0$, uniquely determined $\theta \bmod 2\pi\mathbb{Z}$. We will also use the following facts about exp:

(1) $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for every $z_1, z_2 \in \mathbb{C}$. (This can be gotten from the power series.)
(2) exp is periodic with period $2\pi i$. (This is well known.)
(3) For every $z \in \mathbb{C} \setminus \{0\}$, there is $b \in \mathbb{C}$ such that $\exp(b) = z$. (Write $z = r\exp(i\theta)$ with $r \in (0,\infty)$ and $\theta \in \mathbb{R}$. Then take $b = \log(r) + i\theta$, using the usual definition of $\log\colon (0,\infty) \to \mathbb{R}$.)

We first claim that if Ω is a connected set such that $\exp(\Omega) \subset B_r(a)$, then $\exp|_\Omega$ is injective. Let $T = \{-ta\colon t \in [0,\infty)$. Write $a = r\exp(i\theta)$ with $r \in (0,\infty)$ and $\theta \in \mathbb{R}$. Then $\exp(z) \in T$ if and only if $\operatorname{Im}(z) \in \theta + 2\pi\mathbb{Z}$. (This is easily checked using the facts above.) The connected components of

$$\{z \in \mathbb{C}\colon \operatorname{Im}(z) \in \theta + 2\pi\mathbb{Z}\}$$

are the strips

$$S_n = \{z \in \mathbb{C} \colon \theta + 2\pi n < \mathrm{Im}(z) < \theta + 2\pi(n+1)\}$$

for $n \in \mathbb{Z}$. Since $T \cap B_r(a) = \varnothing$, there is $n \in \mathbb{Z}$ such that $\Omega \subset S_n$. One easily checks (again using the facts above) that $\exp|_{S_n}$ is injective. In particular, $\exp|_{\Omega}$ is injective.

It also follows that if $\Omega \subset \mathbb{C}$ is one region such that $\exp(\Omega) = B_r(a)$, then the collection of all such regions is

$$\{2\pi i n + \Omega \colon n \in \mathbb{Z}\}.$$

Now we prove the existence of Ω and the formulas involving the function log. Choose any $b \in \mathbb{C}$ such that $\exp(b) = a$. Theorem 10.14 of Rudin provides a holomorphic function $g_0 \colon B_r(a) \to \mathbb{C}$ such that $g_0'(z) = z^{-1}$ for all $z \in B_r(a)$. Set $g(z) = g_0(z) - g_0(a) + b$. Then also $g'(z) = z^{-1}$ for all $z \in B_r(a)$. Furthermore $g(a) = b$.

Define a holomorphic function $h \colon B_r(a) \to \mathbb{C}$ by

$$h(z) = \frac{\exp(g(z))}{z}.$$

(Note that $0 \notin B_r(a)$.) Using $g'(z) = z^{-1}$, we get

$$h'(z) = \frac{\exp(g(z))g'(z)z - \exp(g(z))}{z} = 0$$

for all $z \in B_r(a)$. It follows that h is constant. (Consider the form the power series for h must take.) Check that $h(a) = 1$. It follows that $\exp(g(z)) = z$ for all $z \in B_r(a)$. We can now take $\Omega = g(B_r(a))$, which is open by the Open Mapping Theorem and connected because g is continuous and $B_r(a)$ is connected. This proves the existence of Ω. Moreover, $g \colon B_r(a) \to \Omega$ is surjective by definition and injective because $\exp(g(z)) = z$ for all $z \in B_r(a)$. Therefore g is the inverse of $\exp|_{\Omega}$, which shows that the derivative of the inverse of $\exp|_{\Omega}$ is z^{-1}.

It remains only to compute the series in the case $a = 1$. Whenever $w \in \mathbb{C}$ with $|w| < 1$, we have

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}.$$

Putting $w = 1 - z$, we get, for $|z - 1| < 1$,

$$\frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n.$$

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

has radius of convergence 1 and its term by term derivative is the series for z^{-1}. Therefore it differs from $g(z)$ be a constant. Taking the number b with $\exp(b) = 1$ to be $b = 0$, we find that the constant is zero by comparing the values at 0. Therefore

$$\log(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

when $|z - 1| < 1$. $\square$

The proof given above is not what most students have done. Instead, most people have proceeded as follows. One uses the formula $\exp(x+iy) = e^x(\cos(y) + i\sin(y))$ to find a region Ω such that exp defines a bijection from Ω to $B_1(1)$. In terms of the standard real version of the logarithm function (also denoted log), the explicit description of Ω (not really needed to solve the problem) is

$$\Omega = \left\{ x + iy \colon x, y \in \mathbb{R},\ -\tfrac{\pi}{2} < y < \tfrac{\pi}{2}, \text{ and } x < \log(2\cos(y)) \right\}.$$

One then uses the Inverse Function Theorem to show that $\log = (\exp|_{\Omega})^{-1}$ is holomorphic, and one differentiates the equation $\exp(\log(z)) = z$ to determine its derivative.