Little proofreading has been done. Some parts of problems have several different solutions. The solutions are written to be independent of each other, so they can be put in assignments independently. In particular, the same residue computation lemma appears in all three solutions.

1. **Problem 1: Rudin, Chapter 10, Problem 8**

This problem is counts as two ordinary problems.

**Problem 1.1.** Let \( P \) and \( Q \) be polynomials such that \( \deg(Q) \geq \deg(P) + 2 \). Let \( R \) be the rational function \( R(z) = \frac{P(z)}{Q(z)} \) for \( z \in \mathbb{C} \) such that \( Q(z) \neq 0 \).

1. Prove that \( \int_{-\infty}^{\infty} R(x) \, dx \) is equal to \( 2\pi i \) times the sum of the residues of \( R \) in the upper half plane. (Replace the integral over \([-A, A]\) by the integral over a suitable semicircle, and apply the Residue Theorem.)
2. What is the analogous statement for the lower half plane?
3. Use this method to compute \( \int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx \).

It is convenient to begin the solution with a lemma.

**Lemma 1.2.** Let \( p \) be a polynomial of degree \( n \). Then there exist constants \( m_p, M_p, r_p > 0 \) such that for all \( z \in \mathbb{C} \) with \( |z| \geq r_p \), we have \( m_p |z|^n \leq |p(z)| \leq M_p |z|^n \).

We give a direct proof below. But one can also derive this lemma by showing, using algebraic properties of limits, that if \( p(z) = \sum_{k=0}^{n} a_k z^k \) for \( z \in \mathbb{C} \), then

\[
\lim_{|z| \to \infty} \frac{p(z)}{z^n} = \lim_{|z| \to \infty} \sum_{k=0}^{n} a_k z^{k-n} = a_n.
\]

**Proof of Lemma 1.2.** There are \( a_0, a_1, \ldots, a_n \in \mathbb{C} \), with \( a_n \neq 0 \), such that \( p(z) = \sum_{k=0}^{n} a_k z^k \) for all \( z \in \mathbb{C} \). Define

\[
m_p = \frac{|a_n|}{2}, \quad M_p = \sum_{k=0}^{n} |a_k|, \quad \text{and} \quad r_p = \max \left( 1, \frac{2}{|a_n|} \sum_{k=0}^{n-1} |a_k| \right).
\]

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Let $z \in \mathbb{C}$ satisfy $|z| \geq r_p$. Then, using $|z| \geq 1$ at the second step and $r_p \geq \frac{2}{|a_n|} \sum_{k=0}^{n-1} |a_k|$ at the fourth step,
\[
|p(z)| \geq |a_n| \cdot |z|^n - \sum_{k=0}^{n-1} |a_k| \cdot |z|^k \geq |a_n| \cdot |z|^n - |z|^{n-1} \sum_{k=0}^{n-1} |a_k|
\]
\[
\geq |a_n| \cdot |z|^n - r_p^{-1} |z|^n \sum_{k=0}^{n-1} |a_k| \geq |a_n| \cdot |z|^n - \left(\frac{|a_n|}{2}\right) |z|^n = m_p |z|^n.
\]

Also, using $|z| \geq 1$ at the second step,
\[
|p(z)| \leq \sum_{k=0}^{n} |a_k| |z|^k \leq |z|^n \sum_{k=0}^{n} |a_k| = M_p |z|^n.
\]

This completes the proof. □

Solution to part (1). For $A > 0$, we define curves $\gamma_A$, $\rho_A$, and $\sigma_A$ in $\mathbb{C}$ by $\gamma_A(t) = t$ for $t \in [-A, A]$, $\rho_A(t) = Ae^{it}$ for $t \in [0, \pi]$, and $\sigma_A(t) = Ae^{it}$ for $t \in [\pi, 2\pi]$. Then $\gamma_A + \rho_A$, $\gamma_A - \sigma_A$, and $\rho_A + \sigma_A$ are cycles.

We further let $Z_+$ be the set of $z$ in the upper half plane such that $Q(z) = 0$, and we let $Z_-$ be the set of $z$ in the lower half plane such that $Q(z) = 0$. Thus $Z_+$ and $Z_-$ are finite sets. Let $m_p, M_p, r_p, m_Q, M_Q, r_Q$ be the constants of Lemma 1.2 for the polynomials $P$ and $Q$. Also set $L = \sup_{z \in Z_+ \cup Z_-} |z|$.

We first claim that if $z \in Z_-$ and $A > L$, then $\text{Ind}_{\gamma_A + \rho_A}(z) = 0$. Indeed, the path $t \mapsto z - it$, for $t \in [0, \infty)$, does not intersect $\text{Ran}(\gamma_A + \rho_A)$, so $z$ is in the unbounded component of $\text{Ran}(\gamma_A + \rho_A)$.

We next claim that if $z \in Z_+$ and $A > L$, then $\text{Ind}_{\gamma_A + \rho_A}(z) = 1$. Indeed, by Theorem 10.11 of Rudin, we know that $\text{Ind}_{\rho_A + \sigma_A}(z) = 1$, since $\rho_A + \sigma_A$ is essentially the circle of radius $A$ and center 0, and $|z| < A$. Moreover, consideration of the path $t \mapsto z + it$, for $t \in [0, \infty)$, which does not intersect $\text{Ran}(\gamma_A - \sigma_A)$, shows that $z$ is in the unbounded component of $\text{Ran}(\gamma_A - \sigma_A)$. Thus $\text{Ind}_{\gamma_A - \sigma_A}(z) = 0$. Since integration of a fixed function is additive in the chains over which one is integrating, it follows that
\[
\text{Ind}_{\gamma_A + \rho_A}(z) = \text{Ind}_{\gamma_A - \sigma_A}(z) + \text{Ind}_{\gamma_A + \rho_A}(z) = 1.
\]

The claim is proved.

The Residue Theorem now implies that if $A > L$ then
\[
(1) \quad \int_{-A}^{A} R(x) \, dx = 2\pi i \sum_{z \in Z_+} \text{Res}(R; z) - \int_{\rho_A} R(z) \, dz.
\]

We now claim that $\lim_{A \to \infty} \int_{\rho_A} R(z) \, dz = 0$. For $A \geq \max(r_p, r_Q)$, we have, using the choices of $m_Q$ and $M_P$ and the estimates from Lemma 1.2,
\[
(2) \quad \left| \int_{\rho_A} R(z) \, dz \right| = \left| \int_{0}^{\pi} P(Ae^{-it})iAe^{-it} \, dt \right| \leq \int_{0}^{\pi} \frac{|P(Ae^{-it})|Ae^{-it}|}{|Q(Ae^{-it})|} \, dt
\]
\[
\leq \int_{0}^{\pi} M_PA^{d(P)+1} \frac{m_P}{m_QA^{d(Q)}A^{d(P)-d(Q)+1}} \, dt \leq \left(\frac{\pi M_P}{m_Q}\right) A^{d(P)-d(Q)+1}.
\]

Since $\deg(P) - \deg(Q) + 1 < 0$, the claim follows.
Substituting the claim into (1), we deduce that $\lim_{A \to \infty} \int_{-A}^{A} R(x) \, dx$ exists and is equal to $2\pi i \sum_{z \in \mathbb{Z}_+} \text{Res}(R; z)$. \(\Box\)

It isn’t sufficient to prove that $\lim_{z \to \infty} R(z) = 0$. Knowing this sets one up to use the Dominated Convergence Theorem, but one must still produce a dominating function.

It is easy to use Lemma 1.2 to prove directly that the function $R$ is Lebesgue integrable on $(-\infty, \infty)$. It is not hard to compute the relevant winding numbers using Theorem 10.37 of Rudin. But some justification does need to be given.

Solution to part (2) (sketch). Let the notation be the same as in the solution to part (1). Methods similar to those used there show that if $A > L$ then $\text{Ind}_{\gamma_A} R(z) = 1$ for all $z \in \mathbb{Z}_+ \cup \mathbb{Z}_-$. Therefore $\int_{-\infty}^{\infty} R(x) \, dx = -2\pi i \sum_{z \in \mathbb{Z}_-} \text{Res}(R; z)$. \(\Box\)

Instead of repeating all the work, one can reduce part (2) to part (1).

Second solution to part (2). Let the notation be the same as in the solution to part (1). We claim that $\lim_{A \to \infty} \int_{\gamma_A} R(z) \, dz = 0$. For $A > L$, by Theorem 10.11 of Rudin we have $\text{Ind}_{\gamma_A} R(z) = 1$ for all $z \in \mathbb{Z}_+ \cup \mathbb{Z}_-$. Therefore

$$\int_{\gamma_A} R(z) \, dz = 2\pi i \sum_{z \in \mathbb{Z}_+ \cup \mathbb{Z}_-} \text{Res}(R; z).$$

Combining this fact with the claim, we get

$$2\pi i \sum_{z \in \mathbb{Z}_+ \cup \mathbb{Z}_-} \text{Res}(R; z) = 0.$$

Therefore

$$\lim_{A \to \infty} \int_{-A}^{A} R(x) \, dx = 2\pi i \sum_{z \in \mathbb{Z}_+} \text{Res}(R; z) = -2\pi i \sum_{z \in \mathbb{Z}_-} \text{Res}(R; z).$$

This completes the proof. \(\Box\)

The following lemma is convenient for the computation of the residues needed in part (3).
Lemma 1.3. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let $f$ be a holomorphic function on $\Omega \setminus \{a\}$ which has a simple pole at $a$. Then $\text{Res}(f; a) = \lim_{z \to a} (z - a)f(z)$.

Proof. Since $f$ has a simple pole at $a$, by definition there are $c \in \mathbb{C} \setminus \{0\}$ and a holomorphic function $g$ on $\Omega$ such that

$$f(z) = g(z) + \frac{c}{z - a}$$

for all $z \in \Omega \setminus \{a\}$. Moreover, by definition, $\text{Res}(f; a) = c$. Now

$$\lim_{z \to a}(z - a)f(z) = \lim_{z \to a} ((z - a)g(z) + c) = 0 \cdot g(a) + c = c.$$

This completes the proof. $\square$

Solution to part (3). Set $\omega = \exp(\pi i/4)$. Then

$$1 + z^4 = (z - \omega)(z - \omega^3)(z - \omega^5)(z - \omega^7).$$

So the function $R(z) = \frac{z^2}{1 + z^4}$ has two poles in the upper half plane, namely simple poles at $\omega$ and at $\omega^3$. By part (1) and Lemma 3.2, we therefore have, factoring out powers of $\omega$ and repeatedly using $\omega^2 = i$ at the fourth step,

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} dx = 2\pi i \left( \text{Res}(R; \omega) + \text{Res}(R; \omega^3) \right)$$

$$= 2\pi i \left( \lim_{z \to \omega} (z - \omega)R(z) + \lim_{z \to \omega^3} (z - \omega^3)R(z) \right)$$

$$= 2\pi i \left( \frac{\omega^2}{(\omega - \omega^3)(\omega - \omega^5)(\omega - \omega^7)} + \frac{\omega^6}{(\omega^3 - \omega)(\omega^3 - \omega^5)(\omega^3 - \omega^7)} \right)$$

$$= 2\pi i \left( \frac{\omega^{-1}}{(1 - i)(1 - (-1))} + \frac{\omega^{-3}}{(1 - (-i))(1 - i)(1 - (-1))} \right)$$

$$= \left( \frac{\pi i}{2} \right)(\omega^{-1} + \omega^{-3}) = \left( \frac{\pi i}{2} \right)(-i\sqrt{2}) = \frac{\pi}{\sqrt{2}}.$$

This completes the solution. $\square$

2. Problem 2: Rudin, Chapter 10, Problem 11

Problem 2.1. Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| \neq 1$. Calculate

$$\int_0^{2\pi} \frac{1}{1 - 2\alpha \cos(\theta) + \alpha^2} d\theta$$

by integrating $(z - \alpha)^{-1}(z - 1/\alpha)^{-1}$ around the unit circle.

We will use the following lemma to compute residues. (This has also been used previously. It isn’t in Chapter 10 of Rudin’s book, but it was proved in class.) For the residues needed in this problem, a different calculation is given in Remark 2.3.

Lemma 2.2. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let $f$ be a holomorphic function on $\Omega \setminus \{a\}$ which has a simple pole at $a$. Then $\text{Res}(f; a) = \lim_{z \to a} (z - a)f(z)$.
Proof. Since \( f \) has a simple pole at \( a \), by definition there are \( c \in \mathbb{C} \setminus \{0\} \) and a holomorphic function \( g \) on \( \Omega \) such that
\[
f(z) = g(z) + \frac{c}{z-a}
\]
for all \( z \in \Omega \setminus \{a\} \). Moreover, by definition, \( \text{Res}(f; a) = c \). Now
\[
\lim_{z \to a} (z-a)f(z) = \lim_{z \to a} ((z-a)g(z) + c) = 0 \cdot g(a) + c = c.
\]
This completes the proof. \( \square \)

Solution. Define a closed curve \( \gamma \) in \( \mathbb{C} \) by
\[
\gamma(\theta) = e^{i\theta} \quad \text{for} \quad \theta \in [0, 2\pi].
\]
Define a meromorphic function \( f_\alpha \) on \( \mathbb{C} \) by
\[
f_\alpha(z) = \frac{1}{(z-\alpha)(z-\frac{1}{\alpha})}.
\]
Then \( f_\alpha \) has simple poles at \( \alpha \) and at \( \alpha^{-1} \).

We have
\[
\int_\gamma f(z) \, dz = \int_0^{2\pi} \frac{1}{(e^{i\theta}-\alpha)(e^{i\theta}-\frac{1}{\alpha})}ie^{i\theta} \, d\theta
\]
\[
= \int_0^{2\pi} \frac{-i\alpha}{(e^{i\theta}-\alpha)(e^{-i\theta}-\alpha)} \, d\theta = \int_0^{2\pi} \frac{1}{1-2\alpha\cos(\theta)+\alpha^2} \, d\theta.
\]
We now compute this integral by the residue theorem.

Suppose \( |\alpha| < 1 \). Then \( \text{Ind}_\gamma (\alpha) = 1 \) and \( \text{Ind}_\gamma (1/\alpha) = 0 \) by Theorem 10.11 of Rudin. Lemma 2.2 gives
\[
\text{Res}(f_\alpha, \alpha) = \frac{1}{\alpha - \frac{1}{\alpha}} = \frac{\alpha}{\alpha^2 - 1}.
\]
Therefore
\[
\int_0^{2\pi} \frac{1}{1-2\alpha\cos(\theta)+\alpha^2} \, d\theta = \left( \frac{1}{-i\alpha} \right) \int_\gamma f_\alpha(z) \, dz
\]
\[
= \left( \frac{1}{-i\alpha} \right) 2\pi i \text{Res}(f_\alpha, \alpha) = -\frac{2\pi}{\alpha^2 - 1}.
\]
Suppose now \( |\alpha| > 1 \). Then, using the result for \( 1/\alpha \) at the second step, we have
\[
\int_0^{2\pi} \frac{1}{1-2\alpha\cos(\theta)+\alpha^2} \, d\theta = \int_0^{2\pi} \frac{\alpha^{-2}}{1-2\alpha^{-1}\cos(\theta)+\alpha^{-2}} \, d\theta
\]
\[
= \frac{2\pi\alpha^{-2}}{\alpha^{-2} - 1} = \frac{2\pi}{\alpha^2 - 1}.
\]
This completes the solution. \( \square \)

**Remark 2.3.** The residues
\[
\text{Res}(f_\alpha, \alpha) = \frac{\alpha}{\alpha^2 - 1} \quad \text{and} \quad \text{Res}(f_\alpha, \alpha^{-1}) = -\frac{\alpha}{\alpha^2 - 1}
\]
can be read directly off the partial fraction decomposition
\[
f_\alpha(z) = \left( \frac{\alpha}{\alpha^2 - 1} \right) \left( \frac{1}{z-\alpha} - \frac{1}{z-\alpha^{-1}} \right),
\]
without the need for Lemma 2.2.
3. Problem 3: Rudin, Chapter 10, Problem 13

The following problem is Problem 13 in Chapter 10 of Rudin’s book.

Problem 3.1. Prove that
\[ \int_0^\infty \frac{1}{1 + x^n} \, dx = \frac{\pi/n}{\sin(\pi/n)} \]
for \( n \in \mathbb{Z}_{>0} \) with \( n \geq 2 \).

Hint: Use a path from 0 to \( R \) to \( R \exp(2\pi i/n) \) to 0.

The following lemma is convenient for the computation of the residues needed here. It isn’t in Chapter 10 of Rudin’s book, but it was proved in class.

Lemma 3.2. Let \( \Omega \subset \mathbb{C} \) be an open set, let \( a \in \Omega \), and let \( f \) be a holomorphic function on \( \Omega \setminus \{a\} \) which has a simple pole at \( a \). Then \( \text{Res}(f; a) = \lim_{z \to a} (z - a) f(z) \).

Proof. Since \( f \) has a simple pole at \( a \), by definition there are \( c \in \mathbb{C} \setminus \{0\} \) and a holomorphic function \( g \) on \( \Omega \) such that
\[ f(z) = g(z) + \frac{c}{z - a} \]
for all \( z \in \Omega \setminus \{a\} \). Moreover, by definition, \( \text{Res}(f; a) = c \). Now
\[ \lim_{z \to a} (z - a) f(z) = \lim_{z \to a} ((z - a) g(z) + c) = 0 \cdot g(a) + c = c. \]
This completes the proof. \( \square \)

Solution. Set \( \omega = \exp(\pi i/n) \). For \( r \in (1, \infty) \), define paths \( \rho_r, \sigma_r : [0, r] \to \mathbb{C} \) by \( \rho_r(t) = t \) and \( \sigma_r(t) = t \omega^2 \) for \( t \in [0, r] \). Also define \( \gamma_r : [0, 2\pi/n] \to \mathbb{C} \) and \( \beta_r : [2\pi/n, 2\pi] \to \mathbb{C} \) by \( \gamma_r(t) = r e^{it} \) for \( t \in [0, 2\pi/n] \) and \( \beta_r(t) = r e^{it} \) for \( t \in [2\pi/n, 2\pi] \). Then \( \gamma_r + \beta_r, \rho_r + \gamma_r - \sigma_r \), and \( \sigma_r + \beta_r - \rho_r \) are cycles.

The formula
\[ f(z) = \frac{1}{1 + z^n} \]
defines a meromorphic function on \( \mathbb{C} \), with poles at \( \omega, \omega^3, \ldots, \omega^{2n-1} \).

Using \( r > 1 \), we get \( \text{Ind}_{\gamma_r + \beta_r}(\omega) = 1 \) by Theorem 10.11 of Rudin. Also, the path \( t \mapsto t \omega \), for \( t \in [1, \infty) \), is continuous, goes to \( \infty \) as \( t \to \infty \), and has range disjoint from \( \text{Ran}(\sigma_r + \beta_r - \rho_r) \), so \( \text{Ind}_{\sigma_r + \beta_r - \rho_r}(\omega) = 0 \). Therefore
\[ \text{Ind}_{\rho_r + \gamma_r - \sigma_r}(\omega) = \text{Ind}_{\gamma_r + \beta_r}(\omega) - \text{Ind}_{\sigma_r + \beta_r - \rho_r}(\omega) = 1. \]
On the other hand, for \( k = 2, 3, \ldots, n \), the path \( t \mapsto t \omega^k \), for \( t \in [1, \infty) \), is continuous, goes to \( \infty \) as \( t \to \infty \), and has range disjoint from \( \text{Ran}(\rho_r + \gamma_r - \sigma_r) \). So \( \text{Ind}_{\rho_r + \gamma_r - \sigma_r}(\omega^k) = 0 \). We can now apply the Residue Theorem using the cycle \( \rho_r + \gamma_r - \sigma_r \). The condition \( \text{Ind}_{\rho_r + \gamma_r - \sigma_r}(z) = 0 \) for \( z \notin \mathbb{C} \) is vacuous, so we get
\[ \int_{\rho_r + \gamma_r - \sigma_r} f(z) \, dz = 2\pi i \text{Res}(f; \omega). \]
Since \( n \geq 2 \),
\[ \lim_{r \to \infty} \int_{\rho_r} f(z) \, dz = \lim_{r \to \infty} \int_0^r \frac{1}{1 + t^n} \, dt = \int_0^\infty \frac{1}{1 + t^n} \, dt \]
exists and is finite. Similarly
\[
\lim_{r \to \infty} \int_{\sigma_r} f(z) \, dz = \lim_{r \to \infty} \int_0^r \frac{1}{1 + (\omega^2 t)^n} \omega^2 \, dt = \omega^2 \int_0^{\infty} \frac{1}{1 + t^n} \, dt.
\]
We claim that
\[
\lim_{r \to \infty} \int_{\gamma_r} f(z) \, dz = 0.
\]
For \(|z| > 2\), we have
\[
\left| \frac{1}{1 + z^n} \right| \leq \frac{1}{|z|^n - 1} \leq \frac{2}{|z|^n},
\]
so for \(r > 2\) we have, since the length of \(\gamma_r\) is \(2\pi r/n\),
\[
\left| \int_{\gamma_r} f(z) \, dz \right| \leq \left( \frac{2\pi r}{n} \right) \left( \frac{2}{r^n} \right) = \frac{4\pi}{nr^{n-1}}.
\]
Since \(n \geq 2\), the claim follows. So
\[
(1 - \omega^2) \int_0^{\infty} \frac{1}{1 + t^n} \, dt = 2\pi i \text{Res}(f; \omega).
\]
We next calculate \(\text{Res}(f; \omega)\). We use Lemma 3.2. We have, using \(\omega^n = -1\) at the second step,
\[
\text{Res}(f; \omega) = \lim_{z \to \omega} \frac{z - \omega}{z^n + 1} = \lim_{z \to \omega} \frac{\omega z - \omega}{(\omega z)^n + 1} = -\omega \lim_{z \to 1} \frac{z - 1}{z^n - 1}
\]
\[
= -\omega \lim_{z \to 1} \frac{1}{z^{n-1} + z^{n-2} + \ldots + 1} = -\frac{\omega}{n}.
\]
We conclude
\[
\int_0^{\infty} \frac{1}{1 + t^n} \, dt = \frac{2\pi i \text{Res}(f; \omega)}{1 - \omega^2} = -\frac{2\pi i \omega}{n(1 - \omega^2)}
\]
\[
= \frac{2\pi i}{n(\omega - \omega^{-1})} = \frac{\pi/n}{(\omega - \omega^{-1})} = \frac{\pi/n}{\sin(\pi/n)}.
\]
This completes the proof. \(\square\)