For a in $L^2(\mathbb{R})$, consider the 2π periodic torus $\mathbb{T}$. Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{T}$, then $\exp(i \alpha \pi)$ is an exponential function.

Let $a \in C_c(\mathbb{R})$, then $\mathbb{R}$ is with respect to $\mathbb{T}$. Let's consider $\lim_{n \to \infty} \int_{\mathbb{T}} f(x) \exp(i \alpha \pi \frac{x}{n}) \, dx$.

Some sense shows that $\sum_{n=-\infty}^{\infty} a(n) \exp(i \alpha \pi \frac{x}{n})$ is an orthogonal system in $L^2(\mathbb{T})$.

For $f \in L^2([0, 2 \pi])$, define $f_n(x) = \frac{1}{2\pi} \int_{x-n}^{x+n} f(t) \exp(i \alpha \pi \frac{t}{n}) \, dt$.

Now let's suppose $a \in L^2(\mathbb{R})$, then the sum converges uniformly, so $\hat{a}$ is a function in $L^2(\mathbb{T})$.

If $a \in L^2(\mathbb{R})$, then convergence is not in $L^2(\mathbb{T})$ but a.e.

Now, consider $a \in L^2(\mathbb{R})$, then convergence is not in $L^2(\mathbb{T})$ but a.e.

Verify that for $x \in \mathbb{R}$:

$$\hat{a}(x) = \int_{\mathbb{R}} a(t) e^{-2 \pi i xt} \, dt.$$  This is the continuous Fourier transform.

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A picture of such a form:

An ox: better in some ways (here technically awkward):

\[ h_n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2} \]

What will happen

Set \( H_n(t) = H(t/n) \), \( h_n(x) = n h_n(nx) \).

Then we get \( (H_n) \) is \( C_0 \) approximating for \( C_0(\mathbb{R}) \), that is, for all \( g \in C_0(\mathbb{R}) \)

\[ \lim_{n \to \infty} \| H_n g - g \|_\infty = 0 \]

Also \( (h_n) \) is \( C_0 \) approximating for \( L^1(\mathbb{R}) \).

If \( \| f - h_n \|_1 \to 0 \) as \( n \to \infty \), the \( H_n \) will be:

Lemma 1: \( (x) \Rightarrow h_n = H_n \), that is, \( h_n(x) = \int_{\mathbb{R}} H_n(t) e^{itx} \, dt \).

Proof: Change of variables

Lemma 2: \( (x) \Rightarrow \forall p \in [1, \infty) \), \( \forall f \in L^p(\mathbb{R}) \), \( \| f - h_n \|_p \to 0 \) as \( n \to \infty \).

We can show \( p = 1, p = 2 \). Explicitly: if \( \| f \|_p \) is finite, then \( \| f - h_n \|_p \) [not proved]

Theorem: if \( h \in \mathcal{L} \):

\[ \int_{\mathbb{R}} h(t) \, dt = 1 \]

If \( g \in C_0 \), say, continuous, then

\[ \int_{\mathbb{R}} g(x) \, h_n(x) \, dx = \int_{\mathbb{R}} g(x) \, h_n(x) \, dx \]

\[ \Rightarrow \int_{\mathbb{R}} g(x) \, h_n(x) \, dx \to 0 \]

Since \( g \) not close to 0

The case \( \mathcal{L} \) was from normal space in \( L^1(\mathbb{R}) \).

The choice here is \( H \circ \frac{1}{n} \).

One can check: \( h_n(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \)

The choice here is \( H(x) = e^{-x^2} \)