Theorem 2 \( \forall f \in L^p(R), \| f \|_p \rightarrow \infty \) as \( n \rightarrow \infty \).

We need to check: \( f \ast h_n \) is even defined. Only \( p = 1 \) was done.

We will need only \( p = 1, p = 2 \) here, and for unrolled. Use only \( p = 1 \).

General fact: we will use most of the proof here: \( \text{If } f \in L^p, g \in L^q \text{ then } f \ast g \text{ is } L^r \text{ with } \| f \|_p \| g \|_q \leq \| f \ast g \|_r \leq \| f \|_p \| g \|_q. \)

If \( f \) is measurable and \( h \) is \( L^1 \), then \( f \ast h \) is \( L^1 \), and \( \| f \ast h \|_1 \leq \| f \|_1 \| h \|_1 \). If \( f \) is a.e. measurable and \( h \) is \( L^1 \), then \( f \ast h \) is a.e. measurable.

Claim \( f \in L^1(\mathbb{R}) \Rightarrow \left( \int_{\mathbb{R}} |f(y)h_n(x-y)| \, d\lambda(y) \right)^p \leq \int_{\mathbb{R}} |f(y)|^p \int_{\mathbb{R}} |h_n(x-y)| \, d\lambda(y) \). If \( f \in L^1(\mathbb{R}) \), then the claim \( f \ast h_n \) is a.e. measurable. If \( f \) is a.e. measurable, then \( f \ast h_n \) is a.e. measurable.

Proof of claim: For non-negative \( f \), use non-negative Fatou's.

(Remain assume \( f \) is \( L^1 \)) So, \( f \ast h_n \) is non-negative, \( L^1 \), but may be infinite in places.

Proving two claims show \( \int_{\mathbb{R}} (f \ast h_n)^p \, d\lambda \leq \left( \int_{\mathbb{R}} f(y)^p h_n(x-y) \, d\lambda(y) \right)^p \leq \| f \|_p^p \| h_n \|_p^p \). In particular, \( f \ast h_n \) is finite a.e.

For general \( f \), consider the pos and neg parts of real and imag parts.

Turning & convolution gives \( L^p \) a.e. and \( \| f \ast h_n \| \rightarrow \infty \), a.e.

So the appropriate lin combo of these is stable & defined a.e. Claim proved.
Claim \( f \in L_p(\mathbb{R}) \Rightarrow \| f * h_n - f \|_p = \int_{\mathbb{R}} |f(y) - f(x)| h_n(y) \, d\mu(y) \).

[Did not say before that \( f \in L_p(\mathbb{R}) \) but that follows from the claim, since integration right is finite: \( h_n \leq 1 \) and \( \| \cdot \|_p \) is bounded.]

Proof: Change variables: \( f * h_n (x) = \int_{\mathbb{R}} f(x-y) h_n(y) \, d\mu(y) \), \( \| f \|_p = \int_{\mathbb{R}} |f(x)|^p \, d\mu(x) \).

Also write \( f(x) = \int_{\mathbb{R}} f(x+y) h_n(y) \, d\mu(y) \).

\( \| f * h_n - f \|_p = \int_{\mathbb{R}} \| f(x-y) - f(x) \|_p h_n(y) \, d\mu(y) \).

\leq \left( \int_{\mathbb{R}} \| f(x-y) - f(x) \|_p h_n(y) \, d\mu(y) \right)^p \quad \text{Use Jensen again}

\leq \int_{\mathbb{R}} \| f(x-y) - f(x) \|_p h_n(y) \, d\mu(y).

Now integrate w.r.t. \( x \):

\( \int_{\mathbb{R}} \| f * h_n(x) - f(x) \|_p \, d\mu(x) \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \| f(x-y) - f(x) \|_p h_n(y) \, d\mu(y) \right) \, d\mu(x) \).

Using monotone Fubini:

\( \int_{\mathbb{R}} \| f * h_n(x) - f(x) \|_p \, d\mu(x) \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \| f(x-y) - f(x) \|_p h_n(y) \, d\mu(y) \right) \, d\mu(x) \).

Claim proved.

Recall Jensen: \( \mu \) is prob measure, \( f \) convex

\[ \ln \mu \text{ is concave up}. \]

Then \( \mu \left( \int_{\mathbb{R}} f(x) \, d\mu(x) \right) \leq \int_{\mathbb{R}} \mu(f(x)) \, d\mu(x). \) \( \mu \) is concave up. \( \| f \|_p = \int_{\mathbb{R}} |f(x)|^p \, d\mu(x) \).

Recall exp for convexity of \( \mu \):

Let \( s, t \in \text{domain}(\mu) \). Then \( \mu(s + t \delta) \leq \mu(s) \delta + \mu(t) \).

\( \mu(1: \delta^2) = 1 - \delta, \mu(1: \delta^2) = \delta \). \( \mu \) is concave up. Then Jensen gives this.

Need prob measure.
Claim: \[ \| f + \varepsilon h_n - f \|_p \leq \int_{\mathbb{R}} \| T_y(f) - f \|_p \, h_n(y) \, d\bar{m}(y) \]

Proof result: want \( \| f + \varepsilon h_n - f \|_p < \varepsilon \).

Since \( h \in L^1(\mathbb{R}) \) there is \( M \) such that \( \int_{\mathbb{R} \setminus [-M, M]} h_n \, d\bar{m} < \frac{\varepsilon}{2 \cdot 2^p (\| f \|_p + 1)} \).

Now recall \( y \mapsto T_y f \) is cont. So \( y \mapsto \| T_y f - f \|_p \) is cont. and zero at \( y = 0 \). So \( \exists \delta > 0 \) s.t. \( |y| < \delta \Rightarrow \| T_y f - f \|_p < \frac{\varepsilon}{2^p} \).

Choose \( N \) s.t. \( n > N \Rightarrow \frac{M}{n} < \delta \). Then \( \int_{\mathbb{R} \setminus [-M, M]} h_n \, d\bar{m} = \int_{\mathbb{R} \setminus [-M, M]} h \, d\bar{m} < \frac{\varepsilon}{2^p (\| f \|_p + 1)} \),

change of var.