Math 618

\[ \Omega \subset \mathbb{C} \] open, \( f : \Omega \to \mathbb{C}, \ z_0 \in \Omega, \) and \( f'(z_0) \) exists.

Convert to two real vars: \( f_1(x,y) = \text{Re}(f(x+iy)), \ f_2(x,y) = \text{Im}(f(x+iy)) \).

Then \( f(x+iy) = f_1(x,y) + \text{i} f_2(x,y) \).

Write sum difference quotients \( \frac{f(z_0+h) - f(z_0)}{h} \). Take first \( h = \epsilon e^{-\epsilon} \) for suitable \( \epsilon > 0 \). We get

\[ \frac{f_1(x, y) - f_1(x_0, y_0) + \text{i} f_2(x, y) - f_2(x_0, y_0)}{\epsilon} \to f'(z_0). \]

Take \( \text{Re}(\cdot), \text{Im}(\cdot) : \Omega \to \mathbb{R}^2 \).

\[ \partial_1 f_1(x_0, y_0) + \text{i} \partial_2 f_1(x_0, y_0) = f'(z_0) \]

\[ \partial_1 f_2(x_0, y_0) = \partial_2 f_1(x_0, y_0) \quad \text{and} \quad \partial_2 f_2(x_0, y_0) = -\partial_1 f_1(x_0, y_0) \]

This is not in Rudin. The eqns (1) are called the Cauchy–Riemann eqns.

If \( f \) is c.d. in real sense (more than just existence of partial deriv) this converse is also true. C–R eqns \( \Rightarrow \) c.d. at \( z_0 \).

\[ f'(z_0) = \text{Re}(z_0), \quad f(x+iy) = x, \quad f'(z_0) = 0. \]

So \( \partial_1 f_1(x_0, y_0) = 0, \quad \partial_2 f_2(x_0, y_0) = 0. \) C–R not satisfied, and \( f \) is nowhere c.d. diff.

Another version of the same idea. If \( f \) is c.d. at \( z_0 \), \( f(z_0) \in \mathbb{C} \). If \( f = (f_1, f_2) \) is real diff. at \( (x_0, y_0) \), then the derivative \( \left( \frac{\partial}{\partial z}, \text{where} \right) \) is in \( M_2(\mathbb{C}) \)

Claim \( \text{rank } 1 \mathbb{R}^2 = 1 \mathbb{R}^2 \). \( \text{dim}_{\mathbb{R}} \mathbb{C} = 2 \) but \( \text{dim}_{\mathbb{R}} (M_2(\mathbb{R}^2)) = 4. \)

C–d. diff. is a very rigid condition. We will see that if \( f \) is c.d. diff. in an open set \( \Omega \) then \( f' \) is automatically also c.d. diff. (and v.v.)

Lema: \( f' \) exists if \( f \) is c.d. at \( z_0. \) Pf: \( f \) is in real case. \( \text{use rule.} \)

Used diff rules from real case.

Thus: The usual sum rule, scalar multiple rule, product rule, quotient rule, and chain rule for real diff. hold in \( \mathbb{C} \) case with exactly the same formulas.

Pf: As in real case, the lemma will be needed as in real case “\( \text{as in real case}. \)"
Ex: \( f(0) = 0, f'(x) = 1 \), \( f(x) = \frac{e^x}{x} \) all diff. w/ derivs 0, 0,1. (no int or reducible)

For. Every cx real, \( f \) is cx diff. on domain, with same formulas for deriv as in real case.

Ex: \( z \rightarrow R(b), z \rightarrow \Im(x), z \rightarrow \Re(z) \) are nowhere cx diff. (and plr)

Defn: \( \mathbb{C} \subset \mathbb{R} \). Put \( f \) is representable by power series on \( \Omega \) if \( \forall \alpha \in \mathbb{N}^k \), \( \sum_{n=0}^\infty c_n (z \rightarrow \Omega) \) converges to \( f(z) \).

Defn (no oppy, not in Rabi). \( f: \mathbb{C} \rightarrow \Omega \). Then \( f \) is Weitly representable by power series if \( \forall z_0 \in \mathbb{C}, \forall \Omega \) open w/ \( z_0 \in \Omega \), \( f(z) = \sum_{n=0}^\infty c_n (z \rightarrow \Omega) \) converges to \( f(z) \).

Ex: The series \( \sum_{n=0}^\infty z^n \) converges for all \( z \in \mathbb{C} \) (since converges absolutely)

Set define \( f \) in \( \mathbb{C} \) which is weilly representable by power series (One can shw, by monaplicity of series, \( f(\exp(z)) \) is weil representable by power series in the stronger sense. We will need to do this.)

Thm (on radius of convergence): Let \( c_0, c_1, \ldots \in \mathbb{C} \). Set \( R = \lim_{n \rightarrow \infty} \frac{1}{|c_n|} \). (\( R > 0 \) if the limit is zero; \( R = 0 \) if \( \lim_{n \rightarrow \infty} |c_n| = \infty \)). Then the series \( \sum_{n=0}^\infty c_n (z \rightarrow \Omega) \) is:

1. Converges for \( |z| < R \).
2. Converges \( \forall |z| = R \).
3. Converges uniformly on \( \overline{B_R(z)} \) whenever \( s < R \).

Pf: is same as in real case.

Remarks of ideas: If \( |z-z_0| > R \), then terms of series \( \sum_{n=0}^\infty c_n (z \rightarrow \Omega) \) have norm \( |c_n| |z-z_0|^n \rightarrow 0 \), then \( \sum_{n=0}^\infty c_n (z \rightarrow \Omega) \)

(2) follows from (3). For (3) need to show \( n \) th st. \( |c_n(z)\| \leq M\left( \frac{|z|}{R} \right)^n \forall n \), \( |z| \leq s < R \), and \( \forall \epsilon > 0 \) any \( s \), there is a \( \delta \) such that \( |z| < s + \delta \).

Key point: \( \frac{s}{R} < 1 \). So can use Weierstrass test.

Thm. Under hipt. prov. \( f \). the series \( \sum_{n=0}^\infty c_n (z \rightarrow \Omega) \) has the same radius of convergence. (\( \Omega \) \( \subseteq \mathbb{C} \), \( \Omega \) open. \( f(z) = \sum_{n=0}^\infty c_n (z \rightarrow \Omega) \) is weily representable by power series. Then \( f \) is cx diff. and \( f \) is weily representable by power series.

Cor 1. If \( f: \Omega \rightarrow \Omega \) is weily representable by power series, then \( f \) is cx diff. and \( f \) is weily representable by power series.

Cor 2. If \( f: \Omega \rightarrow \Omega \) weily representable by power series. Then \( \forall n, f^n \) weel representable by power series.

Cor 3. exp is cx diff. and \( \exp' = \exp \).

Cor 4. If \( f \) weily rep. by power series on \( \Omega, \alpha, \beta \in \Omega \). Then the derivatives \( c_n \) of \( f \).

Ex: \( f(z) = \sum_{n=0}^\infty c_n (z \rightarrow \Omega) \) are uniquely determined.

Pf. Differentiate term by term to find \( f^{(n)}(z) = n! c_n, \Delta c_n = \frac{1}{n!} (f^{(n)}(z) \bigr|_{z=0}) \bigr|_{z=0} \).