Math 618

\( f: [0,1] \to \mathbb{C} \) piecewise \( C^1 \), \( f: \text{Ran}(\gamma) \to \mathbb{C} \) continuous. Then \( \int_{\gamma} f(z)dz = \int_0^1 f(x(t)) \gamma'(t) dt \)

Recall: \((X, \mu)\) is mes. space, \( \varphi: X \to [0,1] \) bijective, \( \mathcal{D}(\varphi \gamma) = \gamma \). Then \( f(z) = \int_X \frac{1}{(z - \xi)^d} \mu(d\xi) \) is representable by power series (strong sense).

Remarks: consider piecewise \( C^1 \) reparametrization \( \gamma \), given by a piecewise \( C^1 \) strictly incr or strictly decr bijection \( \varphi: [0,1] \to [0,1] \). Replace \( \gamma \) by \( \varphi \gamma \). Check:

\( \varphi \gamma \) is still piecewise \( C^1 \) (more break points, coming from those of \( \gamma \) and of \( \varphi \)).

Using chain rule, what: if \( \gamma \) is strictly incr then \( \int_{\gamma} f(z)dz = \int_{\gamma} f(\varphi \gamma)(\varphi \gamma'(t)) dt \).

^ "orientation reversing."

Sums:

1) Suppose \( \gamma: [\gamma_1, \gamma_2, \cdots] \to \mathbb{C} \) for \( j = 1, 2, \cdots \), with \( \gamma_j(0) = 0 \). Form concatenation:

\( \gamma: [\gamma_1, \gamma_2, \cdots, \gamma_j, \gamma_{j+1}, \cdots] \to \mathbb{C} \). (It: \( \gamma_1 \) on \([0, p_1]\), and \( \gamma_{j+1} \) on \([p_j, p_{j+1}]\). Then \( \int_{\gamma} f(z)dz = \sum_j \int_{\gamma_j} f(z)dz \).

2) For \( \Omega \subset \mathbb{C} \), let \( C_1(\Omega) \) be the free holomorphic gp on the set of piecewise \( C^1 \) curves in \( \Omega \). [not quite the same as \( C_1(\Omega, \mathbb{C}) \) they used in path, and always sumdmn: std 1-simplex]. If \( f: \Omega \to \mathbb{C} \) is cont., not on \( \partial \Omega \) gp hom \( C_1(\Omega) \otimes \mathbb{C} \to \mathbb{C} \) extending \( \partial \Omega \).

Define length \( \ell(\gamma) \) of \( \gamma \) (curve) as \( \int_{\gamma} |\gamma'(t)| dt \). (It's what you think it should be)

Abs: \( |\int_{\gamma} f(z)dz| \leq \ell(\gamma) \sup_{z \in \text{Ran}(\gamma)} |f(z)| \).

Easy to curve: (1) \( \gamma(t) = r + re^{it} \) on \([0, 2\pi]\), positively oriented circle center \( r \), radius \( c \).

(2) oriented line segment. (3) oriented triangle.

Then let \( \gamma: [\gamma_1, \gamma_2] \to \mathbb{C} \) be piecewise \( C^1 \), and closed \( \gamma(0) = \gamma(1) \). Define, for \( z \in \mathbb{C} \setminus \text{Ran}(\gamma) \), \( \text{Ind}_\gamma(z) = \frac{1}{2\pi i} \lim_{\delta \to 0} \int_{\gamma(\gamma)} \frac{1}{s - z} ds \). Then \( \text{Ind}_\gamma \) is cont., integer valued, and zero on the unbounded component of \( \mathbb{C} \setminus \text{Ran}(\gamma) \).

\( \text{Ran}(\gamma) \) is open, so \( \exists \epsilon > 0 \) s.t. \( \text{Ran}(\gamma) \subset B_{\epsilon}(0) \). Then \( \mathbb{C} \setminus B_{\epsilon}(0) \subset C \setminus \text{Ran}(\gamma) \) and is connected.

Prove \( \text{Ind}_\gamma(z) = \frac{1}{2\pi i} \lim_{\delta \to 0} \int_{\gamma(\gamma)} \frac{-z(x)}{s - z} ds \). This is rop by power series, so cont.

Mirror, \( \text{Im} \text{Ind}_\gamma(z) \) is zero (easy), so if pure \( Z \)-valued, then \( \text{Ind}_\gamma(z) \) must be zero on unbounded component.
Prime \( \mathbb{P} \)-valued. Fix \( z \), and set \( \varphi(b) = \exp \left( \int_0^b \frac{\partial \varphi(s)}{\partial \varphi(s) - z} \, ds \right) \).

Heuristic: \( \varphi(b) = \exp \left( \log (\varphi(b) - z) - \log (\varphi(s) - z) \right) \), with \( \log(\_\_\_) \) chosen to vary continuously with \( b \). (\( \varphi(s) = \text{real} \) but \( \log \) is multiple valued, hence values differ by \( 2\pi i \cdot \text{integer} \).

Claim \( \varphi(z) = 1 \). Proof: Fix \( z \), calculate \( \frac{\varphi'(b)}{\varphi(b) - z} = \frac{\partial \varphi(s)/\partial \varphi(s) - z}{\varphi(b) - z} \), except for \( b \) in a finite set \( S \).

Off \( S \), \( b \to \frac{\varphi(b)}{\varphi(b) - z} \) has a limit (quotient rule) equal to \( \frac{\varphi(0)}{\varphi(0) - z} \)\( \text{ at } b \to 0 \).

Next, \( \varphi(z) = 1 \), and so \( \frac{\varphi(b)}{\varphi(b) - z} = \frac{\varphi(s)}{\varphi(s) - z} \), (\( s \) is constant) = \( \frac{\varphi(0)}{\varphi(0) - z} \).

So \( \varphi(z) = 1 \). Claim proved.

Claim says: \( \exp \left( 2\pi i \log \varphi(z) \right) = 1 \), so \( \log \varphi(z) \) is locally constant on \( \mathbb{C} \setminus \text{Re}(x) \).

Ex: \( \varphi(s) = a + re^{i\theta} \), on \( [0, 2\pi] \). Claim: \( \text{Ind}_\varphi(z) = 1 \) when \( |z| = r \). Ref:

| Enough to do case \( z = 0 \). |
| \( \text{B}_r(z) \) is connected. |
| Here, compute:

\[
\text{Ind}_\varphi(z) = 1 \sum_{k=1}^{2\pi} \left( \lim_{x \to 0} \frac{d}{dx} \varphi(x) \right) (2\pi i (k)) = 1. \]

\[
\text{Thm (Candy's Thm.) We only get this generally after careful work.}
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Let \( \Omega \subset \mathbb{C} \) be open. Let \( \gamma \) be a closed curve in \( \Omega \) (more generally, a "cycle" in \( \mathbb{C} \)).

st. \( \text{Ind}_\gamma(z) = 0 \) \( \forall z \in \Omega \setminus \gamma \). Let \( p \in \overline{\Omega} \). Let \( f: \Omega \to \mathbb{C} \) be cont., and hol. on \( \Omega \setminus \{p\} \). Then \( \int_\gamma f(z) \, dz = 0. \) ['] [f is cut at \( p \).]

Note: (i) We will see that \( \gamma \to f \) hol \( \Rightarrow \) \( f \) hol on \( \Omega \). But this is needed in proof.

(ii) to \( \mathbb{C} \) cont. in \( \mathbb{R}^2 \) thru the Green's Thm: it says \( \int f(z) \, dz \) is the integral over the "inside" of \( \gamma \) of some combination of \( \partial f \) and \( \mathbb{C} \)-Requ.

Imply that comb of \( \partial f \) is zero.

\[
\text{Thm (Candy's Formula.) Same hypotheses, but assuming \( f \) hol on \( \Omega \) (no exception at \( p \).) Then \( \text{Ind}_\gamma(z) = \int_\gamma \frac{f(z)}{z - t} \, dz \) for \( z \in \Omega \setminus \text{Re}(x). \)
\]

Note: If \( \Omega = \mathbb{C} \setminus \{0\} \), \( f(z) = \frac{1}{z} \), \( \gamma \) \( \gamma \) = \( e^{it} \) on \( [0, 2\pi] \). Then

the winding \# hypothesis fails: \( 0 \in \mathbb{C} \setminus \Omega \) but \( \text{Ind}_\gamma(0) = 1 \neq 0. \)