Math 618  
Recall \( Z(f) = \{ z \in \mathbb{C} : f(z) = 0 \} \).

We were proving:

Then \( \Omega \subset \mathbb{C} \) region, \( f: \Omega \to \mathbb{C} \) hol. Then:

1. \( Z(f) = \emptyset \) or all of the following holds: \( Z(f) \) is countable, consists of isolated points, and has no limit points in \( \Omega \).
2. If \( Z(f) \) has no limit points in \( \Omega \), then \( \forall \epsilon > 0, \exists \mu \in \mathbb{Z}_{\geq 0}, g \) hol. in \( \Omega \) s.t. \( g \mu \to 0 \) and \( \forall z \in \Omega, f(z) = (z-a)^{\mu} g(z) \).

We had proved: if \( a \in Z(f) \), \( r > 0 \), \( B_r(a) \subset \Omega \), and there is \( b \in B_r(a) \) s.t. \( f(b) \neq 0 \), then the conclusion of (2) holds: \( \exists \mu \in \mathbb{Z}_{\geq 0}, g \) hol. in \( \Omega \) s.t. \( g \mu \to 0 \) and \( f(z) = (z-a)^{\mu} g(z) \) for all \( z \in \Omega \). Moreover, \( a \) is isolated in \( Z(f) \).

Now suppose there is some \( a \in Z(f) \) and some \( r > 0 \) s.t. \( f \) vanishes in \( B_r(a) \).

Let \( A \) be the set of limit pts in \( \Omega \) of \( Z(f) \). Then \( A \) is closed in \( \Omega \). Claim: \( A \) is open.

Proof of claim: Let \( a \in A \). Choose \( r > 0 \) s.t. \( B_r(a) \subset \Omega \). If there is some \( c \in B_r(a) \) such that \( f(c) \neq 0 \), then by what we already did, \( a \) is an isolated pt. in \( Z(f) \). This contradicts \( a \in A \). So \( f \equiv 0 \) in \( B_r(a) \). \( B_r(a) \subset Z(f) \). All pts of \( B_r(a) \) are limit pts of \( B_r(a) \). So \( B_r(a) \subset \mathbb{C} \). Thus \( A \) is open. Claim proved.

\( \Omega \) connected, \( A \neq \emptyset \implies A = \Omega \). So \( Z(f) = \emptyset \).

Thus, if \( Z(f) \neq \emptyset \), then all zeros of \( f \) are isolated (since the first claim last time applies).

It follows that \( Z(f) \) has no limit pts in \( \Omega \). Also, \( \Omega \) is \( \epsilon \)-open, and any open subset contains at most finitely many pts of \( Z(f) \). So \( Z(f) \) is closed.

For \( \Omega \subset \mathbb{C} \) region, \( f: \Omega \to \mathbb{C} \) hol. Suppose \( B \subset \Omega \), \( D \) has a limit pt in \( \Omega \), and \( f \equiv 0 \) in \( B \).

Prove \( \lim_{z \to D} f(z) = 0 \).

Denote \( \Omega \subset \mathbb{C} \) open, \( D \subset \Omega \), \( f: \Omega \setminus \{ \infty \} \to \mathbb{C} \) hol. We say \( f \) has a singularity at \( a \).

1. \( a \) is removable: \( \exists \epsilon > 0, g: \Omega \to \mathbb{C} \) hol., \( g |_{\Omega \setminus \{ a \}} = f \).
2. \( f \) has a pole of order \( k \): \( \exists \epsilon > 0, \mu \in \mathbb{Z}_{\geq 0}, \zeta \in \mathbb{C} \) with \( \zeta \not\in \Omega \) and \( g: \Omega \to \mathbb{C} \) hol. s.t. \( \forall z \in \Omega \setminus \{ a \}, f(z) = \sum_{k=1}^{\mu} \frac{C_k}{(z-a)^k} + \eta(z) \).

We will show below that \( \mu \) and \( C_k, \ldots, C_1 \) are unique. So we can define the order \( k \) of the pole to be \( \mu \), and the principal part of \( f \) at \( a \) to be \( \sum_{k=1}^{\mu} \frac{C_k}{(z-a)^k} \).

3. \( f \) has an essential singularity at \( a \): \( \forall r > 0, f(B_{r}(a) \setminus \{ a \}) \) is dense in \( \mathbb{C} \).
(1) Every singularity is of exactly one of three types.
(2) If \( \exists \text{nbhd} \ U \ni a \) s.t. \( f \) is bounded on \( U \setminus \{a\} \), then \( a \) is removable.
(3) If \( a \) is a pole, then \( m, c_1, \ldots, c_m \) are unique.
(4) If \( a \) is a pole, then \( \lim_{z \to a} (z-a)^m f(z) = \infty \).

No pt here, but there is a statement stronger than (1): if \( a \) is not removable, not a pole, then
\[ \exists z_0 \in C \text{ st. } \forall r > 0, \ G(B_r(a) \setminus \{z_0\}) \ni \sum_{n=0}^{\infty} c_n (z-a)^n \to -\infty. \]

**Proof (2).** Define \( h : \Delta \to C \) by \( h(z) = \left\{ \begin{array}{ll} (z-a)^2 f(z) & z \neq a \\ 0 & z = a \end{array} \right. \).
Then \( h \) is hol. on \( \Delta \setminus \{2a\} \) and at \( a \), using \( f \) both near \( a \), but \( \lim_{z \to a} h(z) + h(a) = -\infty. \)
Therefore \( h \) is hol. on \( \Delta \) as well. So \( \exists c_0, c_1, \ldots, c_{\infty} \text{ s.t. } h(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \text{ on } \overline{B}_r(a). \)
Now, \( c_0 = h(a) = 0 \) and \( c_1 = h'(a) = 0 \), so \( h(z) = \sum_{n=2}^{\infty} c_n (z-a)^n \text{ on } \overline{B}_r(a). \) Then \( s \) is hol. on \( \overline{B}_r(a) \)
and \( g(z) = h(z) \) for \( z \in \overline{B}_r(a) \). Take \( g(z) = f(z) \) for \( z \in C \setminus \overline{B}_r(a). \) (2) is proved.

For (1): at least one of these kinds. Need to show if \( a \) is not essential, then \( a \) is removable.

is a pole. Not ess. \( \exists r > 0 \text{ st. } B_r(a) \setminus \{a\} \text{ and the set } T = \{ B_r(a) \setminus \{a\} \} \text{ is not dense.} \)

Choose \( b \in C \) and \( s > 0 \text{ s.t. } B_s(b) \cap T = \emptyset. \) Define \( g : B_s(b) \setminus \{b\} \to C \) by \( g(z) = \frac{f(z)-b}{z-a}. \)
Then \( g(z) = h(z) \) (by \( \frac{1}{z} \)). So by (2), \( \exists \exists \ \tilde{a} \text{ s.t. } g(z) \) is hol. on \( \overline{B}_r(a) \setminus \{a\}. \) \( \tilde{a} \text{ must be unique.} \)

If \( g(\tilde{a}) \neq 0, \) then \( f(z) = \frac{1}{g(z)} + b \) on \( B_s(b) \setminus \{\tilde{a}\} \) and RHS is hol. on \( B_s(b). \) So \( a \) is removable.

If \( g(\tilde{a}) = 0, \) we \( g(z) \neq 0 \) on \( B_s(b) \setminus \{\tilde{a}\} \) for this \( m, \) h. hol. on \( B_s(b). \) Set by (2) and \( \sum_{n=2}^{\infty} d_n (z-a)^n = g(z) = (z-a)^m \) \( \exists \exists \exists \) \( m_2, \) h. hol. on \( B_s(b) \) st. \( h(z) \neq 0 \) and \( g(z) = (z-a)^m h(z) \) in \( B_s(b) \setminus \{a\}. \) Hence \( h(z) \) as a power series centred at \( a \) divide by \( (z-a)^m \) to get \( g(z) = \sum_{n=0}^{\infty} d_n (z-a)^n. \)

Put form \( \sum_{n=0}^{m-1} d_n (z-a)^n = \) the principal part, \( \sum_{n=m}^{\infty} d_n (z-a)^n = \text{hol. on } B_s(b). \)
Extend over rest of \( \Delta \) as \( f(z) - \sum_{n=m}^{\infty} d_n (z-a)^n \) to get \( h(z) \) on \( \Delta \) except \( a \text{ as a pole.} \)

(\( \sum_{n=0}^{m-1} d_n (z-a)^n \)) \( \exists \exists \exists \) \( m_2, \) h. hol. on \( B_s(b). \) Then \( \lim_{z \to a} h(z) = \infty. \)

\[ g(z) = \left\{ \begin{array}{ll} f(z) - \sum_{n=m}^{\infty} d_n (z-a)^n & z \in B_s(b) \setminus \{a\} \\ \sum_{n=m}^{\infty} d_n (z-a)^n - z \in B_s(b) \end{array} \right. \]