Math 618

**Theorem (Maximum Modulus Theorem)**: If \( |f(z)| \) is a function on \( \mathbb{C} \) that is holomorphic at \( z_0 \) and \( f(z) \neq 0 \) for all \( z \neq z_0 \), then \( |f(z)| \leq |f(z_0)| \) for all \( z \in \mathbb{C} \). If \( f(z) \neq 0 \) in \( \mathbb{C} \), then \( f(z) \) is constant.

**Idea**: Consider the function \( g(z) = \frac{1}{2\pi} \int_{|\theta|=1} \frac{f(z) - f(a) + r e^{i\theta}}{|z - a + r e^{i\theta}|^2} \, d\theta \). As \( r \to 0^+ \), \( g(z) \) approaches \( f(z) \). If \( |f(z)| > \sup|f| \), then \( |f(z)| \to \infty \) as \( |z - a| \to 0 \).

**Proof**: Write \( f(z) = \sum a_n (z-a)^n \) in \( \mathbb{C} \). Suppose \( |f(z)| > \sup|f| \) for \( z \neq a \). Then \( |f(z)| \to \infty \) as \( |z-a| \to 0 \). This means \( c_n = \limsup_{r \to 0^+} \frac{|a_n|}{r^n} \) exists for \( n \to 0 \). If \( c_n \) exists, then \( f(z) = \sum c_n (z-a)^n \) is a **holomorphic function** on \( \mathbb{C} \) with \( \sum c_n \frac{(z-a)^n}{n!} \).

Then \( f(z) \) is **holomorphic**.

**Note**: We'll need \( f \) to be continuous (not just \( \mathbb{C} \) but \( \mathbb{C} \)).

**Proof**: Assume \( f \) is holomorphic \( \mathbb{C} \). If \( M \leq \sup|f| \) on \( \mathbb{C} \), then \( |f(z)| \leq M \) for all \( z \in \mathbb{C} \).

**Write** \( f(z) = \sum c_n (z-a)^n \) in \( \mathbb{C} \). For \( s \in (0, r) \), we have \( \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} \leq \frac{M^2}{r^2} \sum_{n=0}^{\infty} \frac{(a-r e^{i\theta})^n}{n!} \).

Therefore \( |f(z)| \leq \frac{M^2}{r^2} \). The \( \forall s < r \), so \( f(z) \to f(a) \) by continuity. Take \( r \to 0^+ \); the function \( f(z) \) exists for \( r = 0 \).

Then \( f \) is the limit of something, so \( f \) is holomorphic.

**Proof**: Assume \( f \) is holomorphic \( \mathbb{C} \), \( M \geq \sup|f| \) on \( \mathbb{C} \). Then \( |f(z)| \leq M \).

**Write** \( f(z) = \sum c_n (z-a)^n \) in \( \mathbb{C} \). For \( s \in (0, r) \), we have \( \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} \leq \frac{M^2}{r^2} \sum_{n=0}^{\infty} \frac{(a-r e^{i\theta})^n}{n!} \).

Therefore \( |f(z)| \leq \frac{M^2}{r^2} \). The \( \forall s < r \), so \( f(z) \to f(a) \) by continuity. Take \( r \to 0^+ \); the function \( f(z) \) exists for \( r = 0 \).

Then \( f \) is the limit of something, so \( f \) is holomorphic.

**Proof**: Assume \( f \) is holomorphic \( \mathbb{C} \). Then \( f \) is also uniformly continuous on \( \mathbb{C} \).

**Write** \( f(z) = \sum c_n (z-a)^n \) in \( \mathbb{C} \). Then \( \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} \leq \frac{M^2}{r^2} \sum_{n=0}^{\infty} \frac{(a-r e^{i\theta})^n}{n!} \).

Therefore \( |f(z)| \leq \frac{M^2}{r^2} \). The \( \forall s < r \), so \( f(z) \to f(a) \) by continuity. Take \( r \to 0^+ \); the function \( f(z) \) exists for \( r = 0 \).

Then \( f \) is the limit of something, so \( f \) is holomorphic.

**Proof**: Assume \( f \) is holomorphic \( \mathbb{C} \). Then \( f \) is also uniformly continuous on \( \mathbb{C} \).

**Write** \( f(z) = \sum c_n (z-a)^n \) in \( \mathbb{C} \). Then \( \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} \leq \frac{M^2}{r^2} \sum_{n=0}^{\infty} \frac{(a-r e^{i\theta})^n}{n!} \).

Therefore \( |f(z)| \leq \frac{M^2}{r^2} \). The \( \forall s < r \), so \( f(z) \to f(a) \) by continuity. Take \( r \to 0^+ \); the function \( f(z) \) exists for \( r = 0 \).

Then \( f \) is the limit of something, so \( f \) is holomorphic.

**Proof**: Assume \( f \) is holomorphic \( \mathbb{C} \). Then \( f \) is also uniformly continuous on \( \mathbb{C} \).

**Write** \( f(z) = \sum c_n (z-a)^n \) in \( \mathbb{C} \). Then \( \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} \leq \frac{M^2}{r^2} \sum_{n=0}^{\infty} \frac{(a-r e^{i\theta})^n}{n!} \).

Therefore \( |f(z)| \leq \frac{M^2}{r^2} \). The \( \forall s < r \), so \( f(z) \to f(a) \) by continuity. Take \( r \to 0^+ \); the function \( f(z) \) exists for \( r = 0 \).

Then \( f \) is the limit of something, so \( f \) is holomorphic.
Conv. it denies: Let $k \in \mathbb{N}$ be ft. Choose $\varepsilon > 0$ s.t. $|x + 1| < 2\varepsilon$. Set $p = d(x, \infty)$. Then $|f(x) - f(\infty)|$ is smaller.

Theorem 4. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ on $N$. Now let $z \in K$ be L. Fix $\varepsilon < 2\varepsilon$. Then $
abla f_n(z) - \nabla f(z) = \frac{z}{2} - \frac{z}{2}. \quad \begin{aligned} \sup |f_n'(z) - f'(z)| &< \varepsilon. \end{aligned}$

There are other parts: For ex., using Cauchy's Analysis, the limit $f(x)$, etc. --

Con. $f_n \to f$, and in a short study, $f_n \to f$ is also in a short study.

Recall (4), $D = B(0, \varepsilon)$. Then $A(D) = \{ f \in C(D) : \nabla f \not\equiv 0 \}$ with $\| f \|_\infty$. Then $A(D)$ is complete.

Proof. Show closed in $C(D)$. Suppose $f_n \in A(D)$, and $f_n \to f$ uniformly with $t \in C(D)$. Then $f_n \to f$ uniform is the result. Then $f_n \to f$ is also.

We do not conclude $f' \to f'$ uniform in $D$, only on compact subsets of $D$.

Similar results: $H(0) = \{ f : D \to \mathbb{C} : f \text{ hol}\}$ with $\| f \|_\infty$ is complete.

Can replace $D$ and $B$ with $D \subseteq \overline{D}$ if $D$ open, $\overline{D}$ closed.

For very thin $f$

Then $D \subseteq \overline{D}$ open, $f$ hol on $D$, $z \in D$, $f(z) = 0$. Then $\exists V \subseteq \overline{D}$ with $z \in V$ s.t.

(1) $f/V$ is $\overline{V}$.
(2) $W = f(V)$ is open.
(3) $(f/V)^{-1} : W \to V$ is hol.

Use:

**Lemma:** $D \subseteq \overline{D}$ open, $f$ hol on $D$. Define $g : D \times \Delta \to C$ by $g(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w} & z \not= w \\ f'(z) & z = w \end{cases}$

Then $g$ is continuous.