Math 619

From 1st time: Inverse Function Theorem

\[ \Omega \subset \mathbb{C} \text{ open, } f: \Omega \to \mathbb{C}, \quad f'(z) \neq 0. \text{ Then } \exists \text{ open } V \subset \Omega \text{ with } z, w \in V \text{ s.t.} \]

(i) \( f(V) \) is open. (ii) \( W = f(V) \) is open. (iii) \( (f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \) exists.

Use:

Lemma: \( \Omega \subset \mathbb{C} \text{ open, } f: \Omega \to \mathbb{C} \). Define \( g: \Omega \times \Omega \to \mathbb{C} \) by \( g(z,w) = 1 - \frac{f(z)-f(w)}{f'(z) - f'(w)} \).

Pf: \( g \) is continuous.

Proof:

Check at \( (z, z) \) for \( z \in \mathbb{C} \). Let \( z \in \mathbb{C} \). Use \( \text{cont of } f' \) to show \( f' \neq 0 \) at \( B_r(z) \). Let \( \delta > 0 \) such that \( B_r(z) \cap B_\delta(z) = \emptyset \).

Claim: \( g(z,w) = \frac{1}{f'(z)} \) for all \( w \neq z \). Hence \( \frac{1}{f'(z)} \) is defined for all \( z \in \Omega \). Then \( \frac{1}{f'(z)} \) is defined for all \( z \in \Omega \).

Now, \( |g(z,w)| = \left| 1 - \frac{f(z)-f(w)}{f'(z) - f'(w)} \right| \leq 1 \) for all \( z, w \in \Omega \).

Proof:

Use Lemma 1 to choose \( \varepsilon > 0 \) so that \( \forall z \in \Omega \), has:

1. \( \forall z_1, z_2 \in V, \quad |f(z_1) - f(z_2)| \geq \frac{1}{2} |f'(z_0)| |z_1 - z_2| \).

(Using \( g(z, z) \) in Lemma 1. Choose \( \delta \) so that \( g(z, z) < \frac{1}{2} |f'(z_0)| |z_1 - z_2| \).)

Set \( W = f(V) \).

Claim: \( W \) is open. Pf of claim: Let \( a \in W \). Need \( \delta > 0 \) s.t. \( B_\delta(a) \subset W \). Choose \( \delta > 0 \) so that \( B_\delta(a) \subset f(V) \).

1. \( f(z) \) is open. Existence: \( \exists \delta > 0 \text{ such that } f^{-1} \text{ exists.} \)

Now, \( \delta = \frac{1}{2} |f'(z_0)| |z_1 - z_2| \).

Claim: \( f^{-1}: W \to \Omega \) exists. Pf of claim: \( f^{-1}(w_0) \) exists. Verify \( f^{-1}(w) \) as follows:

Claim: \( f^{-1}: \mathbb{C} \to \mathbb{C} \) exists. Pf of claim: \( f^{-1}(w_0) \) exists.

Thus, \( f^{-1}: \mathbb{C} \to \mathbb{C} \) exists.
Temporary notation: \( p_n(z) = z^n \) for \( z \in \mathbb{C}, \text{ for } n \in 1, 2, \ldots \)

Let \( D \subset \mathbb{C} \) open, \( f: D \to \mathbb{C} \). Prove \( f \) is open if \( U \subset \mathbb{C} \) open \( \Rightarrow f(U) \) open.

Lemma: \( n z \Rightarrow p_n(z) \) open in \( \mathbb{C} \). \( \text{Note: } p_2(z) = z^2 + 1 \text{ is not open, range } \{0, 1\} \).

Proof: Let \( U \subset \mathbb{C} \) be open. Let \( w \in p_n(U) \). Choose \( z \in \mathbb{C} \) s.t. \( p_n(z) = w \). If \( z \neq 0 \) then \( p_n(z) / z \to 0 \), \( z \) is an inverse function term. If \( z = 0 \), then \( r > 1 \). By \( B_r(0) \subset U \). Using polar decomposition, check that \( p_r(B_r(0)) = B_{r\phi}(0) \), a nhbd of \( w = 0 \).

The \( D \subset \mathbb{C} \) region \( f: D \to \mathbb{C} \) hol. and not constant. Let \( z_0 \in D \) and let \( n \) be the order of the zero of \( f - f(z_0) \) at \( z_0 \). Initially \( n \leq 1, 2, \ldots \).

Then \( f \) is open in \( D \). Choose \( V \) nhbd \( \mathbb{C} \) \( V \ni z_0 \to f(V) \subset \mathbb{C} \) hol. and \( n > 0 \), s.t.

1. \( f(z) = f(z_0) + k(z-z_0)^n \forall z \in V \).
2. \( k(z_0) \neq 0 \forall z \in V \).
3. \( k \) is a homeomorphism from \( V \to B_r(0) \).

Lemma: \( f : D \to \mathbb{C} \) region \( f \) is open on \( D \) \( \Rightarrow f \) is constant.

Proof: Enough to consider open balls, \( a \in D \). Consider \( \Delta v \in D \). Really only need continuity. So assume \( \Omega \subset D \subset \mathbb{C} \). \( a, b \in \Omega \). Set \( x(V) = (1 - \beta) \epsilon + \beta \in [0, 1] \).

Now estimate term at

\[
\text{Max Value: } \text{Term sup } |f(\beta(V)) - f(\epsilon(V))| \leq \sup_{\beta \in [0, 1]} |f(\beta(V)) - f(\epsilon(V))| = 0 \quad \text{Length of interval is 1}
\]

Enrich: get from i.v.c.: \( |g(\beta(V)) - g(\epsilon(V))| \leq \sup_{\beta \in [0, 1]} |g(\beta(V)) - g(\epsilon(V))| \leq \sup_{\beta \in [0, 1]} |f(\beta(V)) - f(\epsilon(V))| = 0 \).

So \( f(\beta(V)) - f(\epsilon(V)) = 0 \) \( \epsilon \to \beta \).

Ph: \( f \) is not constant \( \Rightarrow m \in 1, 2, \ldots \). Since zeros are isolated, by using smaller \( \Omega \), \( W(\mathbb{C} \setminus \{f(z_0)\} \text{ is never zero on } \Delta \}, \text{ and } \Delta \text{ is convex.}

\text{Suppose \( f \) is not constant. When } z_0 = 0 \text{ and } f(z_0) = 0 \text{, then } f \text{ is hol in } \Delta \text{.}

\text{Let } f = z^m \text{ on } \Omega, \text{ where } m \neq 0 \text{. Then } f \text{ is never zero on } \Omega \text{.}
The function $\bar{h}$ is holomorphic in $\Omega$, $\Omega$ is convex, so by an argument from pt. of Cauchy's formula.

If $x_0$ in $\partial \Omega$ is a point of a convex set, there is $h_0$, $w \in \Omega$ s.t. $h_0(x) = \frac{\bar{h}(x)}{\bar{h}(w)}$ for fixed $x \in \Omega$.

Now can check: $\int \exp(-h) \cdot g \, dz = 0 \text{ in } \Omega$. So choose const. $c$ s.t. $h = h_0 + c$ s.t.

$\exp(-h) \cdot g = 1$. Then $\exp h = g$. So $\exp(-h) g \to$ const. by Lemma.

Now define $k(z) = z \exp\left(\frac{h(z)}{m}\right)$ for $z \in \Omega$. This gives $f(z) = k(z)^m$ on $\Omega$, which is (i).

Next check $k'(z) = \exp\left(\frac{h(z)}{m}\right) + z \frac{d}{dz} \left(\exp\left(\frac{h(z)}{m}\right)\right)$, so $k'(0) \neq 0$.

At $z = 0$: $\uparrow \neq 0$.

Thus $k'(z) \neq 0$ for $z$ near 0. This gives (ii) for small $V$. For (iii) use inverse function theorem.