Let \( D \subseteq \mathbb{C} \) be open. Then \( f \) is a \textit{meromorphic} function if \( D \) is a \textit{holomorphic} function \( f : D \setminus A \to \mathbb{C} \) for some set \( A \subseteq D \) with no limit points in \( D \), and such that \( f \) has \textit{isolated singularities} on \( A \) (since this is above, except that require the singularities to be poles).

We usually tacitly assume no removable singularities.

**Abuse of language:** A \( A \) is the set of poles [singularities] of \( f \).

**Example:** \( f \) hol on \( D \), \( D \cap \mathbb{R} > 0 \) and \( f \) has no limit. For \( z = 0 \), \( f \) is meromorphic.

\( A \) ead one gives a pole at \( \frac{1}{z} \).

**Note:** A above is ahol. Also, \( A = \emptyset \).

**Temp def:** \( f \) has isolated singularities on \( D \), a pole at the singularities. For \( r > 0 \), \( f \) hol on \( B_{\inf}(0) \), define \( \operatorname{Res}_r(f ; a) = \frac{1}{2\pi i} \int_{|z|=r} f(z)\,dz \) with \( \gamma(t) = a + re^{it} \) on \( [0, 2\pi] \).

**Lemma:** \( r_1, r_2 \) ad defined. Then \( \operatorname{Res}_{r_1}(f ; a) = \operatorname{Res}_{r_2}(f ; a) \).

**Proof:** WLOG \( r_1 < r_2 \). Set \( \Gamma = [\gamma_{r_2} \setminus \gamma_{r_1}] \). \text{Then} \( \text{Ind}_{\Gamma}(f) = \frac{1}{2\pi i} \int_{|z|=r} f(z)\,dz \). \text{Calculate} \( \text{Ind}_{\gamma_{r_1}}(f) \), \( \text{Ind}_{\gamma_{r_2}}(f) \) = use ind. of \( \text{Ind}_{\Gamma} \).

Since \( f \) is hol on an open set containing \( \operatorname{Res}_{r}(f) \) and all \( z \) it, \( \text{Ind}_{\Gamma}(f) = 0 \); general Cauchy Thm says \( \int f(z)\,dz = 0 \).

**Defn:** \( \operatorname{Res}_{r}(f ; a) \) is the common value of \( \operatorname{Res}_{r_1}(f ; a) \) for \( r > 0 \) as temp defn.

**Lemma:** Suppose \( f \) has a pole at \( a \) with principal part \( \sum_{k=1}^{m} \frac{C_k}{(z-a)^k} \). Then \( \operatorname{Res}_{r}(f ; a) = C_1 \).

**Proof:** Choose \( r > 0 \) as in temp defn, \( r > a \). Set \( g(z) = f(z) - \sum_{k=1}^{m} \frac{C_k}{(z-a)^k} \) with the removable singularity filled in. Then \( \int_{\gamma_{r}} g(z)\,dz = 0 \) by Cauchy’s Thm. Also check \( \int_{\gamma_{r}} \frac{dz}{(z-a)^{k+1}} = 0 \).

There is an analog for an ess. sing.: the “principal part” is now a series \( \sum_{k=1}^{\infty} \frac{C_k}{(z-a)^k} \) which converges unit on a set in \( B_{\inf}(a) \) so still get \( \operatorname{Res}_{r}(f ; a) = C_1 \).

**Lemma:** (not in Rudin) Suppose \( f \) has a simple pole \( \text{at } a \). Then

\[ \operatorname{Res}_{r}(f ; a) = \lim_{z \to a} (z - a) f(z). \]

**Proof:** Write \( f(z) = g(z) + \frac{C_1}{z-a} \) with \( g \) hol in a nbhd of \( a \). Then check \( \lim_{z \to a} (z-a) g(z) = 0 \)

and \( \lim_{z \to a} \frac{C_1}{(z-a)^{k+1}} = C_k \).

It pole is order 2 or more, or if have an ess sing., then \( \lim_{z \to a} (z-a) f(z) \) does not exist.
\textbf{Residue Theorem:} \( \Omega \subset \mathbb{C} \) open, \( f \) has isolated singular points in \( \Omega \), sing with \( A \). Suppose \( \Gamma \) is a cycle in \( \Omega \setminus A \).

\( \text{Ind}_f(2) = 0 \quad \forall 2 \in \Omega \setminus A \). Then \( S = \{ a \in A : \text{Ind}_f(a) \neq 0 \} \) is finite, and

\[
\frac{1}{2\pi i} \sum_{a \in A} \text{Res}(f, a) = \sum_{a \in A} \text{Ind}_f(a) \text{ Res}(f, a).
\]

\textbf{Notes:}
1. \( \text{Ind}_f(2) = 0 \quad \forall 2 \in \Omega \setminus A \).

2. If \( \text{Ind}_f(a) \neq 0 \quad \forall a \in A \), this is Cauchy's Theorem.

\textbf{Proof:}
Define \( K = \text{Re}(\Gamma) \cup \{ z \in \mathbb{C} : \text{Ind}_f(z) \neq 0 \} \). Then \( K \subset \Omega \) by hypothesis. Also \( \mathbb{C} \setminus K \) is the union of some of the components of \( (\Omega \setminus \text{Re}(\Gamma)) \), including the unbounded compact. So \( K \) is closed a boldd, i.e.,

Write \( S = \{ a_1, a_2, \ldots, a_m \} \) where \( a_1, a_2, \ldots, a_m \) distinct.

Define \( \gamma_j(t) = a_j + r_j e^{it} \quad (0, 2\pi) \) with \( r_j > 0 \). \( B_{r_j}(a_j) \subset (\Omega \setminus A) \cup \gamma_j \).

Also require \( B_{r_j}(a_j) \) disjoint for \( j = 1, \ldots, m \).

Take \( \Gamma_0 = \Gamma - \sum_{j=1}^{m} \text{Ind}_f(a_j) \gamma_j \).

This is a cycle in \( \Omega \setminus A \).

If \( \gamma \notin \Omega \), then \( \text{Ind}_f(2) = 0 \) by hyp. and

\[
\text{Ind}_f(2) = 0 \quad \text{since } B_{r_0}(a_0) \text{ is convex and in } \Omega.
\]

Thus \( \text{Ind}_f(2) = 0 \quad \forall 2 \in \Omega \setminus A \).

If \( \gamma \notin S \) then by construction \( \text{Ind}_f(2) = 0 \).

If \( \gamma \in \Omega \setminus \text{Re}(\Gamma) \) by def. if \( S, \text{Ind}_f(2) = 0 \) and, since \( \gamma \notin B_{r_0}(a_0), \text{Ind}_f(2) = 0 \).

Therefore \( \text{Ind}_f(2) = 0 \quad \forall 2 \notin \Omega \setminus A \), and \( f \) holm. in \( \Omega \setminus A \), so

Using \( \frac{1}{2\pi i} \sum_{a \in A} f(a) = \text{Res}(f, a) \), get result. \( \Box \)