MATH 618 (SPRING 2024, PHILLIPS): HOMEWORK 4

The following problem should be considered to be an example for Rudin, Chapter 10, Problem 25, which was in a previous homework set. However, feel free to use any correct method to solve it (with proof).

Problem 1 (Problem 21 in Chapter 10 of Rudin's book)**.** We want to expand the function

$$f(z) = \frac{1}{1-z^2} + \frac{1}{3-z}$$

as a series of the form $\sum_{n=-\infty}^{\infty} c_n z^n$.

How many such expansions are there? In which region is each of them valid? Find the coefficients c_n explicitly for each of these expansions.

Problem 2 (Problem 19 in Chapter 10 of Rudin's book)**.** Let f and g be holomorphic functions on $B_1(0)$, suppose that $f(z) \neq 0$ and $g(z) \neq 0$ for all $z \in B_1(0)$, and suppose that

$$\frac{f'\left(\frac{1}{n}\right)}{f\left(\frac{1}{n}\right)} = \frac{g'\left(\frac{1}{n}\right)}{g\left(\frac{1}{n}\right)}$$

for all $n \in \mathbb{Z}_{>0}$ with $n > 1$. Find and prove another simple relation between f and g.

The next problem counts as 1.5 ordinary problems.

Problem 3. Let $\Omega \subset \mathbb{C}$ be open. Recall the following.

(1) We define $C_1^{(1)}(\Omega)$ to be the free abelian group on the set of piecewise C^1 curves in Ω, with the element of $C_1^{(1)}(\Omega)$ corresponding to $\gamma \colon [\alpha, \beta] \to \Omega$ being written $[\gamma]$.
(2) If $\Gamma = \sum_{k=1}^{n} m_k [\gamma_k] \in C_1^{(1)}(\Omega)$, with $\gamma_1, \gamma_2, \ldots, \gamma_n$ distinct and $m_1, m_2, \ldots, m_n \in \mathbb{Z} \setminus \{0\}$, then $\operatorname{Ran}(\Gamma) = \bigcup_{k=1}^{n} \operatorname{Ran}(\gamma_k)$.
(3) We define $C_0(\Omega)$ to be the free abelian group on Ω (not to be confused with the algebra of continuous functions on Ω which vanish at infinity). For $z \in \Omega$, we write $[z]$ for the corresponding element of $C_0(\Omega)$.
(4) The homomorphism $\partial \colon C_1^{(1)}(\Omega) \to C_0(\Omega)$ is the abelian group extension of the map sending $[\gamma]$, for $\gamma \colon [\alpha, \beta] \to \Omega$, to $[\gamma(\beta)] - [\gamma(\alpha)]$.
(5) An element $\Gamma \in C_1^{(1)}(\Omega)$ is a *cycle* if $\partial(\Gamma) = 0$.
(6) An element $\Gamma \in C_1^{(1)}(\Omega)$ is an *elementary cycle* if there are $n \in \mathbb{Z}_{\geq 0}$ and piecewise C^1 curves $\gamma_k \colon [\alpha_k, \beta_k] \to \Omega$, for $k = 1, 2, \ldots, n$ such that $\Gamma = \sum_{j=1}^{n} [\gamma_j]$ and:

$$\gamma_2(\alpha_2) = \gamma_1(\beta_1), \quad \gamma_3(\alpha_3) = \gamma_2(\beta_2), \quad \ldots,$$
$$\gamma_n(\alpha_n) = \gamma_{n-1}(\beta_{n-1}), \qquad \text{and} \qquad \gamma_1(\alpha_1) = \gamma_n(\beta_n).$$

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Prove (this counts as one ordinary problem) that for every cycle $\Gamma \in C_1^{(1)}(\Omega)$ there are elementary cycles $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \in C_1^{(1)}(\Omega)$ such that $\operatorname{Ran}(\Gamma_j) \subset \operatorname{Ran}(\Gamma)$ for $j = 1, 2, \ldots, n$ and $m_1, m_2, \ldots, m_n \in \mathbb{Z}$ such that $\Gamma = \sum_{j=1}^{n} m_k \Gamma_j$.

Further prove (this counts as half an ordinary problem). that if $\Gamma \in C_1^{(1)}(\Omega)$ and $\int_\Gamma f(z)\,dz = 0$ for every continuous function $f\colon \operatorname{Ran}(\Gamma) \to \mathbb{C}$, then Γ is a cycle.

Hint for the first part. Write Γ as a suitable formal integer combination of piecewise C^1 curves. By replacing some of these with orientation reversing reparametrizations, one can reduce to the case in which all the coefficients are strictly positive. This case can be done by induction on the sum of the coefficients. Choose any curve occurring in the sum. If it isn't already closed, there is another curve in the sum which starts at its endpoint. Continue. Eventually the endpoint of the newly chosen curve must be the starting point of one of the curves you already have (but *not* necessarily the first one).

The second part can in fact be done using holomorphic functions defined on the whole complex plane.

The following is a rewording (to be more careful) of Rudin, Chapter 10, Problem 28. Do this problem, but possibly with the modifications suggested afterwards. It counts as 1.5 ordinary problems.

Problem 4 (Problem 28 in Chapter 10 of Rudin's book)**.** Let Γ be a closed curve in the plane (continuous but not necessarily piecewise C^1), with parameter interval $[0, 2\pi]$. Let $\alpha \in \mathbb{C} \setminus \operatorname{Ran}(\Gamma)$. Choose a sequence $(\Gamma_n)_{n \in \mathbb{Z}_{>0}}$ of closed curves given by trigonometric polynomials which converges uniformly to Γ. Show that for all sufficiently large m and n, we have $\operatorname{Ind}_{\Gamma_m}(\alpha) = \operatorname{Ind}_{\Gamma_n}(\alpha)$. Define $\operatorname{Ind}_\Gamma(\alpha)$ to be this common value. Prove that it does not depend on the choice of the sequence $(\Gamma_n)_{n \in \mathbb{Z}_{>0}}$. Prove that Lemma 10.39 now holds for closed curves which are merely continuous. Use this result to prove that Theorem 10.40 holds for closed curves which are merely continuous.

The problem says to use trigonometric polynomials for the approximation, but feel free to use piecewise linear functions instead, or some other convenient approximation. Furthermore, it is probably better not to use sequences, despite the statement of the problem. (Of course, don't use Theorem 10.40 of Rudin, but you will want Lemma 10.39.)

For reference, here are the statements of Lemma 10.39 and Theorem 10.40.

Lemma 1. Let $\Gamma_0, \Gamma_1\colon [0, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in $\mathbb{C}$. Let $\alpha \in \mathbb{C}$. Suppose that

$$|\Gamma_1(t) - \Gamma_0(t)| < |\alpha - \Gamma_0(t)|$$

for all $t \in [0, 2\pi]$. Then $\operatorname{Ind}_{\Gamma_0}(\alpha) = \operatorname{Ind}_{\Gamma_1}(\alpha)$.

Theorem 2. Let $\Omega \subset \mathbb{C}$ be open, and let $\Gamma_0, \Gamma_1\colon [0, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in Ω which are homotopic in Ω. Let $\alpha \in \mathbb{C} \setminus \Omega$. Then $\operatorname{Ind}_{\Gamma_0}(\alpha) = \operatorname{Ind}_{\Gamma_1}(\alpha)$.