MATH 618 (SPRING 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 5

This assignment is due on Canvas on Wednesday 8 May 2024 at 9:00 pm.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

Solutions are written to be read independently. Arguments used in more than one solution are therefore repeated in each one.

Problem 1 (Problem 12 in Chapter 10 of Rudin's book)**.** For $t \in \mathbb{R}$, use the Residue Theorem to compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x} \right)^2 e^{itx} \, dx.$$

~~No full solution yet written, but here is an outline. (Also, compare~~ Compare with Rudin Chapter 9 Problem 2. ~~)~~

~~For $a \in (1, \infty)$,~~

Solution (with a few steps just sketched). for $t \in \mathbb{R}$ define ~~paths by $\sigma_a(t) = ae^{it}$ for $t \in [0, \pi]$, $\tau_a(t) = ae^{it}$ for $t \in [\pi, 2\pi]$, $\rho(t) = e^{it}$ for $t \in [\pi, 2\pi]$,~~

$$t_t(z) = \begin{cases} \left(\frac{\sin(z)}{z} \right)^2 e^{itz} & z \in \mathbb{C} \setminus \{0\} \\ 1 & z = 0. \end{cases}$$

Then f_t is an entire function. Also, f_t is integrable on $\mathbb{R}$ because $|f_t(x)| \leq x^{-2}$ when $|x| \geq 1$ and f_t is bounded on $[-1, 1]$ (since $[-1, 1]$ is compact). Further, for $s \in \mathbb{R}$ define $g_s : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g_s(z) = e^{isz}/z^2$. Using the relation

$$[\sin(z)]^2 = \left[\frac{1}{2i} (e^{iz} - e^{-iz}) \right]^2,$$

one checks that

$$f_t(z) = -\left(\frac{1}{4} \right) g_{t+2}(z) + \left(\frac{1}{2} \right) g_t(z) - \left(\frac{1}{4} \right) g_{t-2}(z)$$

for $z \in \mathbb{C} \setminus \{0\}$.

For $a > 1$ define (one of these does not depend on a):

(1) $\sigma_a(\theta) = ae^{i\theta}$ for $\theta \in [0, \pi]$.
(2) $\tau_a(\theta) = ae^{i\theta}$ for $\theta \in [\pi, 2\pi]$.
(3) $\rho(\theta) = e^{i\theta}$ for $\theta \in [0, \pi]$.

Date: 8 May 2024.

(4) $\alpha_a(t) = t$ for $t \in [1, a]$~~, and~~.
(5) $\beta_a(t) = t$ for $t \in [-a, -1]$. ~~Let γ_a be the chain $\alpha_a + \beta_a + \rho$. Set~~

$$f_t(z) = \left(\frac{\sin(z)}{z}\right)^2 e^{itz}$$

(6) $\iota_a(t) = t$ for ~~$z \in \mathbb{C} \setminus \{0\}$, which we can rewrite as~~ $t \in [-a, a]$.

Define $\Gamma_a = [\alpha_a] - [\rho] + [\beta_a]$. Then

$$\Gamma_a + [\sigma_a], \qquad \Gamma_a - [\tau_a], \qquad \text{and} \qquad \Gamma_a - [\iota_a]$$

are immediately seen to be cycles.

For $a > 1$ and $s \in \mathbb{R}$, define

$$\varphi_a(s) = \int_{\Gamma_a} g_s(z)\, dz.$$

We claim that $\psi(s) = \lim_{a\to\infty} \varphi_s(a)$ exists and is given by

$$\psi(s) = \begin{cases} 0 & s > 0 \\ -2\pi s & s < 0. \end{cases}$$

We prove the claim for $s \geq 0$. By considering the negative imaginary axis, one sees that 0 is in the unbounded component of $\mathrm{Ran}(\Gamma_a) \cup \mathrm{Ran}(\sigma_a)$. Therefore $\mathrm{Ind}_{\Gamma_a + [\sigma_a]}(0) = 0$. (Something must be said here.) Using Cauchy's Theorem at the fist step, we then get

$$\begin{aligned} \int_{\Gamma_a} g_s(z)\, dz &= -\int_{\sigma_a} g_s(z)\, dz = -\int_0^\pi \frac{\exp(is\sigma_a(\theta))iae^{i\theta}}{\sigma_a(\theta)^2}\sigma_a'(\theta)\, d\theta \\ &= -\int_0^\pi \frac{\exp(isae^{i\theta})iae^{i\theta}}{(ae^{i\theta})^2}\, d\theta = -\int_0^\pi a^{-1}\exp(isae^{i\theta})iae^{i\theta}\, d\theta. \end{aligned} \tag{1}$$

Since $\sin(\theta) \geq 0$ for $\theta \in [0, \pi]$, and $a, s \geq 0$, we get

$$|\exp(isae^{i\theta})| = |\exp(isa[\cos(\theta) + i\sin(\theta)])| = \exp(-as\sin(\theta)) \leq 1.$$

Therefore the integrand in the last expression in (1) convereges uniformly to 0 as $a \to \infty$. So $\psi(s) = 0$.

To prove the claim for $s < 0$, we first write

$$f_t \int_{\Gamma_a} g_s(z)\, dz = \frac{-e^{(t+2)iz} + 2e^{itz} - e^{(t-2)iz}}{4z^2}. Define \varphi_a \int_{\tau_a} g_s(sz) = dz + \int_{\gamma_a} \frac{e^{isz}}{z^2}{}_{\Gamma_a - \tau_a} g_s(z)\, dz.$$

~~Then~~ One checks (details omitted, but something must be said) that $\mathrm{Ind}_{\Gamma_a - \tau_a}(0) = -1$. The expansion

$$\int_{-a}^{a} f_t g_s(xz)\, dx = -\varphi_a\left(t\frac{1}{z^2} + 2\right)\frac{is}{z} + \varphi_a \frac{(is)^2}{2!} + \frac{(is)^3 z}{3!} + \cdots$$

shows that $\mathrm{Res}(g_s; 0) = is$. Similar methods to the case $s \geq 0$ show that

$$\lim_{a\to\infty} \int_{\tau_a} g_s(tz) - \varphi_a(t-2).\, dz = 0.$$

~~The residue of e^{isz}/z^2 at 0 is is. It should follow (most easily using direct estimation of the integrals of e^{isz}/z^2 over σ_a and τ_a; the Dominated Convergence Theorem should not be needed)that $\lim_{a\to\infty} \varphi_a(s) = -2\pi s$ when $s \geq 0$ and $\lim_{a\to\infty} \varphi_a(s) = 0$ when $s < 0$~~Therefore

$$\lim_{a\to\infty} \int_{\Gamma_a} g_s(z)\, dz = \lim_{a\to\infty} \int_{\tau_a} g_s(z)\, dz + 2\pi i \mathrm{Res}(g_s; 0) = 2\pi s,$$

as desired. The completes the proof of the claim.

With the last step justified because f_t is entire and $\Gamma_a - [\iota_a]$ is a cycle, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\, dx &= \int_{-\infty}^{\infty} f_t(x)\, dx = \lim_{a\to\infty} \int_{-a}^{a} f_t(x)\, dx \\
&= \lim_{a\to\infty} \int_{\iota_a} f_t(x)\, dx = \lim_{a\to\infty} \int_{\Gamma_a} f_t(x)\, dx \\
&= -\left(\frac{1}{4}\right) \varphi(t+2) + \left(\frac{1}{2}\right) \varphi(t) - \left(\frac{1}{4}\right) \varphi(t-2).
\end{aligned}$$

For $t \geq 2$, all terms are zero. For $t \leq -2$, we get

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\, dx = -\left(\frac{1}{4}\right) 2\pi(t+2) + \left(\frac{1}{2}\right) 2\pi t - \left(\frac{1}{4}\right) 2\pi(t-2) = 0.$$

For $t \in [0, 2]$, we get

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\, dx = -\left(\frac{1}{4}\right) \cdot 0 + \left(\frac{1}{2}\right) \cdot 0 - \left(\frac{1}{4}\right) 2\pi(t-2) = \frac{\pi}{2}(2-t).$$

For $t \in [-2, 0]$, we get

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\, dx = -\left(\frac{1}{4}\right) \cdot 0 + \left(\frac{1}{2}\right) 2\pi t - \left(\frac{1}{4}\right) 2\pi(t-2) = \frac{\pi}{2}(t+2).$$

One can put this together in one formula (not necessary):

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\, dx = \frac{\pi}{2}(2 - |t|).$$

This completes the solution. $\square$

The next problem counts as two ordinary problems.

Problem 2 (Problem 8 in Chapter 10 of Rudin's book)**.** Let P and Q be polynomials such that $\deg(Q) \geq \deg(P) + 2$ and $Q(x) \neq 0$ for all $x \in \mathbb{R}$. Let R be the rational function $R(z) = P(z)/Q(z)$ for $z \in \mathbb{C}$ such that $Q(z) \neq 0$.

(1) Prove that $\int_{-\infty}^{\infty} R(x)\,dx$ is equal to $2\pi i$ times the sum of the residues of R in the upper half plane. (Replace the integral over $[-A, A]$ by the integral over a suitable semicircle, and apply the Residue Theorem.)
(2) What is the analogous statement for the lower half plane?
(3) Use this method to compute
$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx.$$

It is convenient to begin the solution with a lemma.

Lemma 1. Let p be a polynomial of degree n. Then there exist constants $m_p, M_p, r_p > 0$ such that for all $z \in \mathbb{C}$ with $|z| \geq r_p$, we have $m_p|z|^n \leq |p(z)| \leq M_p|z|^n$.

We give a direct proof below. But one can also derive this lemma by showing, using algebraic properties of limits, that if $p(z) = \sum_{k=0}^{n} a_k z^k$ for $z \in \mathbb{C}$, then
$$\lim_{|z|\to\infty} \frac{p(z)}{z^n} = \lim_{|z|\to\infty} \sum_{k=0}^{n} a_k z^{k-n} = a_n.$$

Proof of Lemma 1. There are $a_0, a_1, \ldots, a_n \in \mathbb{C}$, with $a_n \neq 0$, such that $p(z) = \sum_{k=0}^{n} a_k z^k$ for all $z \in \mathbb{C}$. Define
$$m_p = \frac{|a_n|}{2}, \qquad M_p = \sum_{k=0}^{n} |a_k|, \qquad \text{and} \qquad r_p = \max\left(1,\ \frac{2}{|a_n|}\sum_{k=0}^{n-1}|a_k|\right).$$
Let $z \in \mathbb{C}$ satisfy $|z| \geq r_p$. Then, using $|z| \geq 1$ at the second step and $r_p \geq \frac{2}{|a_n|}\sum_{k=0}^{n-1}|a_k|$ at the fourth step,
$$\begin{aligned}|p(z)| &\geq |a_n|\cdot|z|^n - \sum_{k=0}^{n-1}|a_k|\cdot|z|^k \geq |a_n|\cdot|z|^n - |z|^{n-1}\sum_{k=0}^{n-1}|a_k| \\ &\geq |a_n|\cdot|z|^n - r_p^{-1}|z|^n\sum_{k=0}^{n-1}|a_k| \geq |a_n|\cdot|z|^n - \left(\frac{|a_n|}{2}\right)|z|^n = m_p|z|^n.\end{aligned}$$
Also, using $|z| \geq 1$ at the second step,
$$|p(z)| \leq \sum_{k=0}^{n}|a_k|\cdot|z|^k \leq |z|^n\sum_{k=0}^{n}|a_k| = M_p|z|^n.$$
This completes the proof. $\square$

Solution to part (1). For $A > 0$, we define curves γ_A, ρ_A, and σ_A in $\mathbb{C}$ by $\gamma_A(t) = t$ for $t \in [-A, A]$, $\rho_A(t) = Ae^{it}$ for $t \in [0, \pi]$, and $\sigma_A(t) = Ae^{it}$ for $t \in [\pi, 2\pi]$. Then ~~$\gamma_A + \rho_A$, $\gamma_A - \sigma_A$, and $\rho_A + \sigma_A$~~ $[\gamma_A] + [\rho_A]$, $[\gamma_A] - [\sigma_A]$, and $[\rho_A] + [\sigma_A]$ are cycles.

We further let Z_+ be the set of z in the upper half plane such that $Q(z) = 0$, and we let Z_- be the set of z in the lower half plane such that $Q(z) = 0$. Thus Z_+ and Z_- are finite sets. Let $m_P, M_P, r_P, m_Q, M_Q, r_Q$ be the constants of Lemma 1 for the polynomials P and Q. Also set $L = \sup_{z \in Z_+ \cup Z_-} |z|$.

We first claim that if $z \in Z_-$ and $A > L$, then $\mathrm{Ind}_{\gamma_A+\rho_A}(z) = 0$. Indeed, the path $t \mapsto z - it$, for $t \in [0, \infty)$, does not intersect $\mathrm{Ran}(\gamma_A + \rho_A)$, so z is in the unbounded component of $\mathrm{Ran}(\gamma_A + \rho_A)$.

We next claim that if $z \in Z_+$ and $A > L$, then $\mathrm{Ind}_{\gamma_A+\rho_A}(z) = 1$. Indeed, by Theorem 10.11 of Rudin, we know that $\mathrm{Ind}_{\rho_A+\sigma_A}(z) = 1$, since $\rho_A+\sigma_A$ is essentially the circle of radius A and center 0, and $|z| < A$. Moreover, consideration of the path $t \mapsto z + it$, for $t \in [0, \infty)$, which does not intersect $\mathrm{Ran}(\gamma_A - \sigma_A)$, shows that z is in the unbounded component of $\mathrm{Ran}(\gamma_A - \sigma_A)$. Thus $\mathrm{Ind}_{\gamma_A - \sigma_A}(z) = 0$. Since integration of a fixed function is additive in the chains over which one is integrating, it follows that

$$\mathrm{Ind}_{\gamma_A + \rho_A}(z) = \mathrm{Ind}_{\gamma_A - \sigma_A}(z) + \mathrm{Ind}_{\gamma_A + \rho_A}(z) = 1.$$

The claim is proved.

The Residue Theorem now implies that if $A > L$ then

$$\int_{-A}^{A} R(x)\, dx = 2\pi i \sum_{z \in Z_+} \mathrm{Res}(R; z) - \int_{\rho_A} R(z)\, dz. \tag{2}$$

We now claim that $\lim_{A \to \infty} \int_{\rho_A} R(z)\, dz = 0$. For $A \geq \max(r_P, r_Q)$, we have, using the choices of m_Q and M_P and the estimates from Lemma 1,

$$\begin{aligned} \left| \int_{\rho_A} R(z)\, dz \right| &= \left| \int_0^{\pi} \frac{P(Ae^{-it}) i A e^{-it}}{Q(Ae^{-it})}\, dt \right| \leq \int_0^{\pi} \frac{|P(Ae^{-it})| |A| |e^{-it}|}{|Q(Ae^{-it})|}\, dt \\ &\leq \int_0^{\pi} \frac{M_P A^{\deg(P)+1}}{m_Q A^{\deg(Q)}}\, dt \leq \left(\frac{\pi M_P}{m_Q} \right) A^{\deg(P) - \deg(Q) + 1}. \end{aligned} \tag{3}$$

Since $\deg(P) - \deg(Q) + 1 < 0$, the claim follows.

Substituting the claim into (2), we deduce that $\lim_{A \to \infty} \int_{-A}^{A} R(x)\, dx$ exists and is equal to $2\pi i \sum_{z \in Z_+} \mathrm{Res}(R; z)$. □

It isn't sufficient to prove that $\lim_{z \to \infty} R(z) = 0$. Knowing this sets one up to use the Dominated Convergence Theorem, but one must still produce a dominating function.

It is easy to use Lemma 1 to prove directly that the function R is Lebesgue integrable on $(-\infty, \infty)$.

It is not hard to compute the relevant winding numbers using Theorem 10.37 of Rudin. But some justification *does* need to be given.

Solution to part (2) (sketch). Let the notation be the same as in the solution to part (1). Methods similar to those used there show that if $A > L$ then $\mathrm{Ind}_{\gamma_A - \sigma_A}(z) = 0$ for $z \in Z_+$, while $\mathrm{Ind}_{\gamma_A - \sigma_A}(z) = -1$ for $z \in Z_-$. So the Residue Theorem gives

$$\int_{-A}^{A} R(x)\, dx = -2\pi i \sum_{z \in Z_-} \mathrm{Res}(R; z) + \int_{\sigma_A} R(z)\, dz.$$

Using the same methods as used to get (3), one shows that $\lim_{A \to \infty} \int_{\sigma_A} R(z)\, dz = 0$. Therefore $\int_{-\infty}^{\infty} R(x)\, dx = -2\pi i \sum_{z \in Z_-} \mathrm{Res}(R; z)$. □

Instead of repeating all the work, one can reduce part (2) to part (1).

Second solution to part (2). Let the notation be the same as in the solution to part (1).

We claim that $\lim_{A\to\infty} \int_{\rho_A+\sigma_A} R(z)\,dz = 0$. For $A \geq \max(r_P, r_Q)$, we have, using the choices of m_Q and M_P and the estimates from Lemma 1,

$$\left|\int_{\rho_A+\sigma_A} R(z)\,dz\right| = \left|\int_0^{2\pi} \frac{P(Ae^{-it})iAe^{-it}}{Q(Ae^{-it})}\,dt\right| \leq \int_0^{2\pi} \frac{|P(Ae^{-it})||A||e^{-it}|}{|Q(Ae^{-it})|}\,dt$$
$$\leq \int_0^{2\pi} \frac{M_P A^{\deg(P)+1}}{m_Q A^{\deg(Q)}}\,dt \leq \left(\frac{2\pi M_P}{m_Q}\right) A^{\deg(P)-\deg(Q)+1}.$$

Since $\deg(P) - \deg(Q) + 1 < 0$, the claim follows.

For $A > L$, by Theorem 10.11 of Rudin we have $\mathrm{Ind}_{\rho_A+\sigma_A}(z) = 1$ for all $z \in Z_+ \cup Z_-$. Therefore

$$\int_{\rho_A+\sigma_A} R(z)\,dz = 2\pi i \sum_{z\in Z_+\cup Z_-} \mathrm{Res}(R; z).$$

Combining this fact with the claim, we get

$$2\pi i \sum_{z\in Z_+\cup Z_-} \mathrm{Res}(R; z) = 0.$$

Therefore

$$\lim_{A\to\infty} \int_{-A}^{A} R(x)\,dx = 2\pi i \sum_{z\in Z_+} \mathrm{Res}(R; z) = -2\pi i \sum_{z\in Z_-} \mathrm{Res}(R; z).$$

This completes the proof. □

The following lemma is convenient for the computation of the residues needed in part (3). It isn't in Chapter 10 of Rudin's book, but it was proved in class this year.

Lemma 2. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$ which has a simple pole at a. Then $\mathrm{Res}(f; a) = \lim_{z\to a}(z-a)f(z)$.

Proof. Since f has a simple pole at a, by definition there are $c \in \mathbb{C} \setminus \{0\}$ and a holomorphic function g on Ω such that

$$f(z) = g(z) + \frac{c}{z-a}$$

for all $z \in \Omega \setminus \{a\}$. Moreover, by definition, $\mathrm{Res}(f; a) = c$. Now

$$\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\big((z-a)g(z) + c\big) = 0 \cdot g(a) + c = c.$$

This completes the proof. □

Solution to part (3). Set $\omega = \exp(\pi i/4)$. Then

$$1 + z^4 = (z-\omega)(z-\omega^3)(z-\omega^5)(z-\omega^7).$$

So the function $R(z) = \frac{z^2}{1+z^4}$ has two poles in the upper half plane, namely simple poles at ω and at ω^3. By part (1) and Lemma 5, we therefore have, factoring out powers of ω and repeatedly using $\omega^2 = i$ at the fourth step,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx &= 2\pi i(\operatorname{Res}(R;\omega) + \operatorname{Res}(R;\omega^3)) \\
&= 2\pi i\left(\lim_{z\to\omega}(z-\omega)R(z) + \lim_{z\to\omega^3}(z-\omega^{3})R(z)\right) \\
&= 2\pi i\left(\frac{\omega^2}{(\omega-\omega^3)(\omega-\omega^5)(\omega-\omega^7)} + \frac{\omega^6}{(\omega^3-\omega)(\omega^3-\omega^5)(\omega^3-\omega^7)}\right) \\
&= 2\pi i\left(\frac{\omega^{-1}}{(1-i)(1-(-1))(1-(-i))} + \frac{\omega^{-3}}{(1-(-i))(1-i)(1-(-1))}\right) \\
&= \left(\frac{\pi i}{2}\right)(\omega^{-1}+\omega^{-3}) = \left(\frac{\pi i}{2}\right)\left(-i\sqrt{2}\right) = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

This completes the solution. $\square$

Alternate residue computation for part (3) (sketch). The residues can be read off directly from the partial fraction decomposition

$$\frac{z^2}{1+z^4} = \frac{1}{4}\left(\frac{\omega^3}{z-\omega} + \frac{\omega^5}{z-\omega^3} + \frac{\omega^7}{z-\omega^5} + \frac{\omega}{z-\omega^7}\right).$$

One can also use a partial fraction decomposition for $(1+z^4)^{-1}$ and multiply it by z^2. $\square$

Problem 3 (Problem 11 in Chapter 10 of Rudin's book)**.** Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| \neq 1$. Calculate

$$\int_0^{2\pi} \frac{1}{1 - 2\alpha\cos(\theta) + \alpha^2}\,d\theta$$

by integrating $(z-\alpha)^{-1}(z-1/\alpha)^{-1}$ around the unit circle.

We will use the following lemma to compute residues. ~~(This has also been used previously.~~ It isn't in Chapter 10 of Rudin's book, but it was proved in class ~~.)~~ this year. For the residues needed in this problem, a different calculation is given in Remark 4.

Lemma 3. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$ which has a simple pole at a. Then $\operatorname{Res}(f;a) = \lim_{z\to a}(z-a)f(z)$.

Proof. Since f has a simple pole at a, by definition there are $c \in \mathbb{C}\setminus\{0\}$ and a holomorphic function g on Ω such that

$$f(z) = g(z) + \frac{c}{z-a}$$

for all $z \in \Omega\setminus\{a\}$. Moreover, by definition, $\operatorname{Res}(f;a) = c$. Now

$$\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\big((z-a)g(z) + c\big) = 0\cdot g(a) + c = c.$$

This completes the proof. $\square$

Solution. Define a closed curve γ in $\mathbb{C}$ by $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Define a meromorphic function f_α on $\mathbb{C}$ by

$$f_\alpha(z) = \frac{1}{(z-\alpha)\left(z-\frac{1}{\alpha}\right)}.$$

Then f_α has simple poles at α and at α^{-1}.

We have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_0^{2\pi} \frac{1}{(e^{i\theta}-\alpha)\left(e^{i\theta}-\frac{1}{\alpha}\right)} ie^{i\theta}\,d\theta \\ &= \int_0^{2\pi} \frac{-i\alpha}{(e^{i\theta}-\alpha)(e^{-i\theta}-\alpha)}\,d\theta = \int_0^{2\pi} \frac{-i\alpha}{1-2\alpha\cos(\theta)+\alpha^2}\,d\theta.\end{aligned}$$

We now compute this integral by the residue theorem.

Suppose $|\alpha| < 1$. Then $\mathrm{Ind}_\gamma(\alpha) = 1$ and $\mathrm{Ind}_\gamma(1/\alpha) = 0$ by Theorem 10.11 of Rudin. Lemma 3 gives

$$\mathrm{Res}(f_\alpha, \alpha) = \frac{1}{\alpha - \frac{1}{\alpha}} = \frac{\alpha}{\alpha^2 - 1}.$$

Therefore

$$\begin{aligned}\int_0^{2\pi} \frac{1}{1-2\alpha\cos(\theta)+\alpha^2}\,d\theta &= \left(\frac{1}{-i\alpha}\right)\int_\gamma f_\alpha(z)\,dz \\ &= \left(\frac{1}{-i\alpha}\right) 2\pi i \mathrm{Res}(f_\alpha,\alpha) = -\frac{2\pi}{\alpha^2-1}.\end{aligned}$$

Suppose now $|\alpha| > 1$. Then, using the result for $1/\alpha$ at the second step, we have

$$\begin{aligned}\int_0^{2\pi} \frac{1}{1-2\alpha\cos(\theta)+\alpha^2}\,d\theta &= \int_0^{2\pi} \frac{\alpha^{-2}}{1-2\alpha^{-1}\cos(\theta)+\alpha^{-2}}\,d\theta \\ &= -\frac{2\pi\alpha^{-2}}{\alpha^{-2}-1} = \frac{2\pi}{\alpha^2-1}.\end{aligned}$$

This completes the solution. □

Remark 4. The residues

$$\mathrm{Res}(f_\alpha,\alpha) = \frac{\alpha}{\alpha^2-1} \qquad \text{and} \qquad \mathrm{Res}(f_\alpha,\alpha^{-1}) = -\frac{\alpha}{\alpha^2-1}$$

can be read directly off the partial fraction decomposition

$$f_\alpha(z) = \left(\frac{\alpha}{\alpha^2-1}\right)\left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha^{-1}}\right),$$

without the need for Lemma 3.

Problem 4 (Problem 13 in Chapter 10 of Rudin's book)**.** Prove that

$$\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi/n}{\sin(\pi/n)}$$

for $n \in \mathbb{Z}_{>0}$ with $n \geq 2$.

The following lemma is convenient for the computation of the residues needed here. It isn't in Chapter 10 of Rudin's book, but it was proved in class this year.

Lemma 5. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$ which has a simple pole at a. Then $\mathrm{Res}(f; a) = \lim_{z\to a}(z-a)f(z)$.

Proof. Since f has a simple pole at a, by definition there are $c \in \mathbb{C}\setminus\{0\}$ and a holomorphic function g on Ω such that

$$f(z) = g(z) + \frac{c}{z-a}$$

for all $z \in \Omega \setminus \{a\}$. Moreover, by definition, $\mathrm{Res}(f; a) = c$. Now

$$\lim_{z \to a} (z - a) f(z) = \lim_{z \to a} \big((z - a) g(z) + c \big) = 0 \cdot g(a) + c = c.$$

This completes the proof. $\square$

Solution. Set $\omega = \exp(\pi i/n)$. For $r \in (1, \infty)$, define paths $\rho_r, \sigma_r \colon [0, r] \to \mathbb{C}$ by $\rho_r(t) = t$ and $\sigma_r(t) = t\omega^2$ for $t \in [0, r]$. Also define $\gamma_r \colon [0, \, 2\pi/n] \to \mathbb{C}$ and $\beta_r \colon [2\pi/n, \, 2\pi] \to \mathbb{C}$ by $\gamma_r(t) = re^{it}$ for $t \in [0, \, 2\pi/n]$ and $\beta_r(t) = re^{it}$ for $t \in [2\pi/n, \, 2\pi]$. Then ~~$\gamma_r + \beta_r$, $\rho_r + \gamma_r - \sigma_r$, and $\sigma_r + \beta_r - \rho_r$~~ $[\gamma_r] + [\beta_r]$, $[\rho_r] + [\gamma_r] - [\sigma_r]$, and $[\sigma_r] + [\beta_r] - [\rho_r]$ are cycles.

The formula

$$f(z) = \frac{1}{1 + z^n}$$

defines a meromorphic function on $\mathbb{C}$, with poles at $\omega, \omega^3, \ldots, \omega^{2n-1}$.

Using $r > 1$, we get $\mathrm{Ind}_{\gamma_r + \beta_r}(\omega) = 1$ by Theorem 10.11 of Rudin. Also, the path $t \mapsto t\omega$, for $t \in [1, \infty)$, is continuous, goes to ∞ as $t \to \infty$, and has range disjoint from $\mathrm{Ran}(\sigma_r + \beta_r - \rho_r)$, so $\mathrm{Ind}_{\sigma_r + \beta_r - \rho_r}(\omega) = 0$. Therefore

$$\mathrm{Ind}_{\rho_r + \gamma_r - \sigma_r}(\omega) = \mathrm{Ind}_{\gamma_r + \beta_r}(\omega) - \mathrm{Ind}_{\sigma_r + \beta_r - \rho_r}(\omega) = 1.$$

On the other hand, for $k = 2, 3, \ldots, n$, the path $t \mapsto t\omega^k$, for $t \in [1, \infty)$, is continuous, goes to ∞ as $t \to \infty$, and has range disjoint from $\mathrm{Ran}(\rho_r + \gamma_r - \sigma_r)$. So $\mathrm{Ind}_{\rho_r + \gamma_r - \sigma_r}(\omega^k) = 0$. We can now apply the Residue Theorem using the cycle $\rho_r + \gamma_r - \sigma_r$. The condition $\mathrm{Ind}_{\rho_r + \gamma_r - \sigma_r}(z) = 0$ for $z \notin \mathbb{C}$ is vacuous, so we get

$$\int_{\rho_r + \gamma_r - \sigma_r} f(z) \, dz = 2\pi i \mathrm{Res}(f; \omega).$$

Since $n \geq 2$,

$$\lim_{r \to \infty} \int_{\rho_r} f(z) \, dz = \lim_{r \to \infty} \int_0^r \frac{1}{1 + t^n} \, dt = \int_0^\infty \frac{1}{1 + t^n} \, dt$$

exists and is finite. Similarly

$$\lim_{r \to \infty} \int_{\sigma_r} f(z) \, dz = \lim_{r \to \infty} \int_0^r \frac{1}{1 + (\omega^2 t)^n} \omega^2 \, dt = \omega^2 \int_0^\infty \frac{1}{1 + t^n} \, dt.$$

We claim that

$$\lim_{r \to \infty} \int_{\gamma_r} f(z) \, dz = 0.$$

For $|z| > 2$, we have

$$\left| \frac{1}{1 + z^n} \right| \leq \frac{1}{|z|^n - 1} \leq \frac{2}{|z|^n},$$

so for $r > 2$ we have, since the length of γ_r is $2\pi r/n$,

$$\left| \int_{\gamma_r} f(z) \, dz \right| \leq \left(\frac{2\pi r}{n} \right) \left(\frac{2}{r^n} \right) = \frac{4\pi}{n r^{n-1}}.$$

Since $n \geq 2$, the claim follows. So

$$(1 - \omega^2) \int_0^\infty \frac{1}{1 + t^n} \, dt = 2\pi i \mathrm{Res}(f; \omega).$$

We next calculate $\mathrm{Res}(f;\omega)$. We use Lemma 5. We have, using $\omega^n = -1$ at the second step,

$$\begin{aligned} \mathrm{Res}(f;\omega) &= \lim_{z\to\omega} \frac{z-\omega}{z^n+1} = \lim_{z\to 1} \frac{\omega z-\omega}{(\omega z)^n+1} = -\omega \lim_{z\to 1} \frac{z-1}{z^n-1} \\ &= -\omega \lim_{z\to 1} \frac{1}{z^{n-1}+z^{n-2}+\cdots+1} = -\frac{\omega}{n}. \end{aligned}$$

We conclude

$$\begin{aligned} \int_0^\infty \frac{1}{1+t^n}\,dt &= \frac{2\pi i \mathrm{Res}(f;\omega)}{1-\omega^2} = -\frac{2\pi i\omega}{n(1-\omega^2)} \\ &= \frac{2\pi i}{n(\omega-\omega^{-1})} = \frac{\pi/n}{(\omega-\omega^{-1})/(2i)} = \frac{\pi/n}{\sin(\pi/n)}. \end{aligned}$$

This completes the proof. □

Alternate residue computation. The residues can be read off directly from the partial fraction decomposition

$$\frac{1}{1+z^n} = -\frac{1}{n}\sum_{k=1}^{n} \frac{\omega^{2k-1}}{z-\omega^{2k-1}}.$$

(This partial fraction decomposition has not been checked.) □