MATH 618 (SPRING 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 6

This assignment is due on Canvas on Wednesday 15 May 2024 at 9:00 pm.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

Problem 1 (Problem 21 in Chapter 10 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be an open set which contains the closed unit disk. Let f be a holomorphic function on Ω such that $|f(z)| < 1$ for all $z \in \mathbb{C}$ such that $|z| = 1$. Determine, with proof, the possible numbers of fixed points of f (that is, solutions to the equation $f(z) = z$) in the open unit disk.

Solution. For $z \in \Omega$, define $g(z) = f(z) - z$ and $h(z) = z$. We apply Rouché's Theorem (Theorem 10.43(b) of Rudin), with $\gamma(t) = \exp(it)$ for $t \in [0, 2\pi]$. Observe that, for $z \in \mathrm{Ran}(\gamma)$, we have

$$|h(z) - g(z)| = |-f(z)| < 1 = |z| = |h(z)|.$$

Moreover, by Theorem 10.11 of Rudin, $\mathrm{Ind}_{\gamma}(z)$ is 0 or 1 for all $z \in \mathbb{C} \setminus \mathrm{Ran}(\gamma)$, and is equal to 1 exactly on the open unit disk. Therefore Rouché's Theorem implies that g and h have the same number of zeros in the open unit disk. Since h has exactly one zero in the open unit disk, so does g. This means that f has exactly one fixed point in the open unit disk. □

Problem 2 (Problem 20 in Chapter 10 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, let $f\colon \Omega \to \mathbb{C}$, and let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on Ω. Suppose that $f_n \to f$ uniformly on compact sets in Ω.

(1) Suppose that, for all $n \in \mathbb{Z}_{>0}$, the function f_n is never zero on Ω. Prove that either $f(z) = 0$ for all $z \in \Omega$ or $f(z) \neq 0$ for all $z \in \Omega$.

(2) If $U \subset \mathbb{C}$ is open and $f_n(\Omega) \subset U$ for all n, prove that f is constant or $f(\Omega) \subset U$.

Solution to (1). Assume that there is $z \in \Omega$ such that $f(z) \neq 0$. Let $z_0 \in \Omega$; we prove $f(z_0) \neq 0$.

First, f is holomorphic by Theorem 10.28 of Rudin. Therefore $\{z \in \Omega\colon f(z) = 0\}$ is countable, by Theorem 10.18 of Rudin. Since there are uncountably many $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$, there is $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$ and such that $f(z) \neq 0$ for all $z \in \partial B_r(z_0)$. Choose $n \in \mathbb{Z}_{>0}$ such that

$$\sup_{z \in \partial B_r(z_0)} |f_n(z) - f(z)| < \inf_{z \in \partial B_r(z_0)} |f(z)|.$$

Since f_n does not vanish on $B_r(z_0)$, it follows from Theorem 10.43(b) of Rudin that f also does not vanish on $B_r(z_0)$. In particular, $f(z_0) \neq 0$. □

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Alternate solution to (1). Assume that there is $z_0 \in \Omega$ such that $f(z_0) = 0$. We prove that $f(z) = 0$ for all $z \in \Omega$.

Choose $r_0 > 0$ such that $B_{r_0}(z_0) \subset \Omega$. Let $0 < r < r_0$. For each n, apply the Maximum Modulus Theorem to $1/f_n$ (see the corollary to Theorem 10.24 of Rudin) to find $\theta_n \in [0, 2\pi]$ such that $|f_n(z_0 + re^{i\theta_n})| \leq |f_n(z_0)|$. Passing to a subsequence of $(f_n)_{n \in \mathbb{Z}_{>0}}$, we may assume that $\theta = \lim_{n \to \infty} \theta_n$ exists.

We claim that $f(z_0 + re^{i\theta}) = 0$. Let $\varepsilon > 0$. Choose N so large that $n \geq N$ implies $|f_n(z) - f(z)| < \frac{1}{3}\varepsilon$ for all $z \in \overline{B_r(z_0)}$. Since f is continuous, we may choose $\delta > 0$ such that $|z - (z_0 + re^{i\theta})| < \delta$ implies $|f(z) - f(z_0 + re^{i\theta})| < \frac{1}{3}\varepsilon$. Choose $n \geq N$ such that $|re^{i\theta_n} - re^{i\theta}| < \delta$. Then, using $|(z_0 + re^{i\theta}) - (z_0 + re^{i\theta_n})| < \delta$ at the first step, and $|f_n(z_0)| = |f_n(z_0) - f(z_0)| < \frac{1}{3}\varepsilon$ at the second step,

$$\begin{aligned} |f(z_0 + re^{i\theta})| &\leq |f(z_0 + re^{i\theta}) - f(z_0 + re^{i\theta_n})| \\ &\qquad + |f(z_0 + re^{i\theta_n}) - f_n(z_0 + re^{i\theta_n})| + |f_n(z_0 + re^{i\theta_n})| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + |f_n(z_0)| < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows that $f(z_0 + re^{i\theta}) = 0$.

We have shown that for every $r \in (0, r_0)$ there is $z \in \Omega$ with $|z - z_0| = r$ such that $f(z) = 0$. Thus, z_0 is a limit point of the set of zeros of f. So $f(z) = 0$ for all $z \in \Omega$. $\square$

Solution to (2). Assume that there is $z_0 \in \Omega$ such that $f(z_0) \notin U$. Let $g_n = f_n - f_n(z_0)$ and let $g = f - f(z_0)$. Then $g_n \to g$ uniformly on compact sets in Ω, and each g_n is never zero on Ω, but $g(z_0) = 0$. The first statement of the problem implies that $g(z) = 0$ for all $z \in \Omega$. Therefore f is constant, with value $f(z_0)$. $\square$

Remark 1. The Open Mapping Theorem does not help with the second statement. All it gives is that if f is not constant, then $f(\Omega) \subset \operatorname{int}(\overline{U})$. In general U is a proper subset of $\operatorname{int}(\overline{U})$, even for connected open subsets of $\mathbb{C}$. For example, if $U = \mathbb{C} \setminus \{0\}$ then $\operatorname{int}(\overline{U}) = \mathbb{C}$. Even requiring U to be simply connected does not help: if $U = \mathbb{C} \setminus [0, \infty)$ then still $\operatorname{int}(\overline{U}) = \mathbb{C}$.

Problem 3. Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $C^\infty(S^1)$, the set of C^∞ functions from the circle S^1 to $\mathbb{C}$. Suppose that for every $m \in \mathbb{Z}_{\geq 0}$, the quantity

$$\sup_{n \in \mathbb{Z}_{>0}} \sup_{t \in S^1} |f_n^{(m)}(t)|$$

is finite. Prove that there are $f \in C^\infty(S^1)$ and a subsequence $(f_{k(n)})_{n \in \mathbb{Z}_{>0}}$ of $(f_n)_{n \in \mathbb{Z}_{>0}}$ such that $f_{k(n)}^{(m)} \to f^{(m)}$ uniformly for every $m \in \mathbb{Z}_{\geq 0}$.

Derivatives of functions on S^1 are to be computed by identifying functions on S^1 with 2π-periodic functions on $\mathbb{R}$ in the usual way.

We will need the following theorem from undergraduate analysis.

Theorem 2. Let $a, b \in \mathbb{R}$ satisfy $sa < b$. Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of C^1 functions on $[a, b]$. Suppose that there are $f, g \in C([a, b])$ such that $f_n \to f$ uniformly and $f_n' \to g$ uniformly. Then f is C^1, $f' = g$, and $f_n' \to f'$ uniformly.

The hypothesis on the convergence of $(f_n)_{n \in \mathbb{Z}_{>0}}$ is overkill. For example, it is enough to require $f_n \to f$ pointwise.

The easiest proof is as follows. Set $k(x) = f(a) + \int_a^x g(t)\,dt$. Since $f_n' \to g$ uniformly and $f_n(a) \to f(a)$, it follows that the functions $x \mapsto f_n(a) + \int_a^x f_n'(t)\,dt$ converge uniformly to k. That is, $f_n \to k$ uniformly. So, on the one hand, $k = f$, while, on the other hand, $k' = g$ by the Fundamental Theorem of Calculus.

Lemma 3. Let $a, b \in \mathbb{R}$ satisfy $a < b$. Let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of differentiable functions on $[a, b]$. Suppose that

$$\sup_{n\in\mathbb{Z}_{>0}} \sup_{x\in[a,b]} |f_n'(x)| < \infty.$$

Then $(f_n)_{n\in\mathbb{Z}_{>0}}$ is uniformly equicontinuous.

Proof. This is an immediate consequence of the Mean Value Theorem, in the form $|f(z) - f(y)| \leq \sup_{x\in[y,z]} |f'(t)|$ whenever $y < z$. □

The form of the Mean Value Theorem used here is true with codomain $\mathbb{C}$, but one can get a worse estimate (still good enough for this purpose) by applying the conventional version to the real and imaginary parts of f. The form here is also valid for functions with values in a Banach space.

Solution. Let $C \subset C([0,\, 2\pi + 1])$ be

$$C = \bigl\{ f \in C([0,\, 2\pi + 1]) \colon f(x + 2\imath) = f(x) \text{ for all } x \in [0, 1] \bigr\},$$

which is a closed subspace of $C([0,\, 2\pi + 1])$ with the usual supremum norm $\|\cdot\|_\infty$. Let V be the set of C^∞ functions in C, a subspace (but not closed).

Define $\varphi \colon C(S^1) \to C$ by $\varphi(f)(t) = f(e^{it})$ for $t \in [0,\, 2\pi + 1]$. Then φ is bijective and isometric. Moreover, by the definitions given in the problem statement, if $f \in C(S^1)$ is in fact C^1, then $\varphi(f') = \varphi(f)'$. Therefore it suffices to do the problem entirely in terms of C and V.

Thus, let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence in V, and suppose that for every $m \in \mathbb{Z}_{\geq 0}$, the quantity

$$\sup_{n\in\mathbb{Z}_{>0}} \sup_{t\in[0,\, 2\pi+1]} |f_n^{(m)}(t)|$$

is finite. We construct by induction on $m \in \mathbb{Z}_{\geq 0}$ functions $g_m \in C$ and strictly increasing functions $j_m, k_m \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $f_{k_m(n)}^{(m)} \to g_m$ uniformly as $n \to \infty$ and $k_m = k_{m-1} \circ j_m$, at $m = 0$ taking $k_{-1}(n) = n$ for all $n \in \mathbb{Z}_{>0}$.

For the base case, the sequence $(f_n)_{n\in\mathbb{Z}_{>0}}$ is uniformly equicontinuous by Lemma 3, so the Arzela-Ascoli Theorem provides $g_0 \in C([0,\, 2\pi + 1])$ and a strictly increasing function $j_0 \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $F_{j_0(n)} \to g_0$ uniformly as $n \to \infty$.

Assume k_m has been constructed. The sequence $\bigl(f_{k_m(n)}^{(m)}\bigr)_{n\in\mathbb{Z}_{>0}}$ is uniformly equicontinuous by Lemma 3, so the Arzela-Ascoli Theorem provides $g_{m+1} \in C([0,\, 2\pi + 1])$ and a strictly increasing function $j_{m+1} \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $f_{k_m(j_{m+1}(n))}^{(m)} \to g_m$ uniformly as $n \to \infty$. We have $g_m \in C$ since C is closed. Set $k_{n+1}(n) = k_m(j_{m+1}(n))$ for $n \in \mathbb{Z}_{>0}$. This completes the induction.

Define $l \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by $l(n) = k_n(n)$ for $n \in \mathbb{Z}_{\geq 0}$.

We claim that l is strictly increasing. Let $n \in \mathbb{Z}_{>0}$. Since $j_{n+1} \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is strictly increasing, we have $j_{n+1}(n + 1) \geq n + 1$. Since k_n is strictly increasing, it follows that $k_n(j_{n+1}(n + 1)) \geq k_n(n + 1) > k_n(n)$. Thus

$$l(n + 1) = k_{n+1}(n + 1) > k_n(n) = l(n).$$

The claim is proved.

We claim that for every $m \in \mathbb{Z}_{\geq 0}$, the sequence $(l(n))_{n \geq m}$ is a subsequence of $(k_m(n))_{n \geq m}$. To prove the claim, since l and k_m are strictly increasing, we need only show that if $n \geq m$ then $l(n) \in \{k_m(r) \colon r \geq m\}$. This is true because

$$\begin{aligned} l(n) &= k_n(n) = k_{n-1}(j_n(n)) = k_{n-2}(j_{n-1}(j_n(n))) \\ &= \cdots = (k_m \circ j_{m+1} \circ j_{m+2} \circ \cdots \circ j_n)(n) \end{aligned}$$

and the functions $j_s \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ are strictly increasing. The claim is proved.

The previous claim implies that for every $m \in \mathbb{Z}_{\geq 0}$, the sequence $(f_{l(n)})_{n \geq m}$ is a subsequence of $(f_{k_m(n)})_{n \geq m}$. Therefore $f_{l(n)}^{(m)} \to g_m$ uniformly as $n \to \infty$. Applying Theorem 2 repeatedly, we see that g_{m-1} is differentiable with $g_{m-1}' = g_m$. Taking $f = g_0$, it follows that for every $m \in \mathbb{Z}_{\geq 0}$, the sequence $(f_{l(n)}^{(m)})$ converges uniformly to $f^{(m)}$. $\square$

Remark 4. Any interval of the form $[0,\, 2\pi + \varepsilon]$, with $\varepsilon > 0$, can be used in place of $[0,\, 2\pi + 1]$. In fact, any interval of length greater than 2π can be used. However, $[0,\, 2\pi]$ does not work. For example, the function $f(t) = t(2\pi - t)$ is C^∞, is in the image in $C[0,\, 2\pi])$ of $C(S^1)$, but is not in the image in $C[0,\, 2\pi])$ of even $C^1(S^1)$.

One can circumvent this by considering only functions f such that $f^{(m)}(0) = f^{(m)}(2\pi)$ (using one sided derivatives), but the solution given is easier to write.

Remark 5. We use $C([0,\, 2\pi + 1])$ and the subspace C instead of $C_{\mathrm{b}}(\mathbb{R})$ and the subspace of 2π-periodic functions because $\mathbb{R}$ is not compact, so that the Arzela-Ascoli Theorem does not directly apply. One can circumvent this with additional argument based on periodicity, but, again, the solution given is easier to write.

Remark 6. The problem was stated in terms of S^1 to avoid issues involving endpoints. However, those issues are really only completely avoided in an argument using $C_{\mathrm{b}}(\mathbb{R})$ and the subspace of 2π-periodic functions.

Problem 4 (Problem 13 in Chapter 14 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of injective holomorphic functions on Ω, and suppose that there is a function $f \colon \Omega \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of Ω. Prove that f is constant or injective. Show by example that both cases can occur.

Solution. We prove the first statement. Assume that $f_n \to f$ uniformly on compact subsets of Ω, and that f is not constant. Then f is holomorphic by Theorem 10.28 of Rudin's book. We need to prove that f is injective, which we do by contradiction. So assume there are $b \in \mathbb{C}$ and distinct $x, y \in \Omega$ such that $f(x) = f(y) = b$.

Since f is not constant and Ω is connected, x and y are isolated zeros of the function $z \mapsto f(z) - b$. Therefore there are $r, s > 0$ such that

$$\overline{B_r(x)} \subset \Omega, \qquad \overline{B_s(y)} \subset \Omega, \qquad \overline{B_r(x)} \cap \overline{B_s(y)} = \varnothing.$$

and $f(z) - b$ is never zero on the set $K = \partial B_r(x) \cup \partial B_s(y)$. Since K is compact, the number $\varepsilon = \inf_{z \in K} |f(z) - b|$ satisfies $\varepsilon > 0$. Again by compactness, there is $n \in \mathbb{Z}_{>0}$ such that $|f_n(z) - f(z)| < \frac{\varepsilon}{2}$ for all $z \in K$. Therefore $\bigl|(f_n(z) - b) - (f(z) - b)\bigr| < \frac{\varepsilon}{2}$ for all $z \in K$. Using Rouché's Theorem on the closed curves $\gamma(t) = re^{it} + x$ and $\sigma(t) = se^{it} + y$, both for $t \in [0, 2\pi]$, we deduce that the functions $z \mapsto f(z) - b$ and $z \mapsto f_n(z) - b$ have the same number of zeros in $B_r(x)$ and also the same number of zeros in $B_s(y)$. Therefore $z \mapsto f_n(z) - b$ has at least one zero in each of $B_r(x)$ and $B_s(y)$, contradicting injectivity of f_n. The first statement is proved.

For the example with f constant, take $\Omega = \{z \in \mathbb{C} \colon |z| < 1\}$, and take $f_n(z) = \frac{1}{n}$ for $n \in \mathbb{Z}_{>0}$ and $z \in \Omega$. Then $(f_n)_{n \in \mathbb{Z}_{>0}}$ converges uniformly on all of Ω to the constant function $f(z) = 0$ for all $z \in \Omega$.

For the example with f injective, take $\Omega = \mathbb{C}$, define $g \colon \Omega \to \mathbb{C}$ by $g(z) = z$ for all $z \in \mathbb{C}$, and take $(f_n)_{n \in \mathbb{Z}_{>0}}$ to be the constant sequence $f_n = g$ for $n \in \mathbb{Z}_{>0}$. Then $(f_n)_{n \in \mathbb{Z}_{>0}}$ converges uniformly to g and g is injective. □

Alternate solution (sketch). We describe a different arrangement of the proof of the first statement.

Assume there are $b \in \mathbb{C}$ and distinct $x, y \in \Omega$ such that $f(x) = f(y) = b$. Replacing f with $z \mapsto f(z) - b$ and f_n with $z \mapsto f_n(z) - f_n(x)$, and using $f_n(x) \to f(x) = b$, we may assume $b = 0$ and $f_n(x) = 0$ for all $n \in \mathbb{Z}_{>0}$. Since f is not constant and Ω is connected, y is an isolated zero of f. Choose $s > 0$ such that $\overline{B_s(y)} \subset \Omega \setminus \{x\}$ and f does not vanish on $\overline{B_s(y)} SM \{y\}$. Using uniform convergence on $\partial B_s(y)$ and Rouché's Theorem, show that there is n such that f_n has a zero in $B_s(y)$. Since $f_n(x) = 0$ as well, this contradicts injectivity of f_n. □

Problem 5 (Problem 15 in Chapter 14 of Rudin's book). Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disk. Let F be the set of holomorphic functions $f \colon D \to \mathbb{C}$ such that $\mathrm{Re}(f(z)) > 0$ for all $z \in D$ and $f(0) = 1$. Prove that F is a normal family.

Can the condition "$f(0) = 1$" be omitted?

Can the condition "$f(0) = 1$" be replaced with "$|f(0)| \leq 1$"?

In Rudin's book, the set D is called U.

Some of the solution has not yet been written.

For convenience, in this problem only, let P be the open right half plane, $P = \{z \in \mathbb{C} \colon \mathrm{Re}(z) > 0\}$.

Lemma 7. Let X be a topological space, let Y and Z be metric spaces, let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of continuous functions $f_n \colon X \to Y$, let $f \colon X \to Y$ be continuous, and let $g \colon Y \to Z$ be continuous. Suppose that $f_n \to f$ uniformly on compact sets in X. Then $g \circ f_n \to g \circ f$ uniformly on compact sets in X.

The assumption that f is continuous is not redundant: on topological spaces which are not "compactly generated", this assumption does not follow from the other hypotheses. However, this assumption is probably not necessary.

Proof of Lemma 7. It is clearly enough to assume that X is compact and that $f_n \to f$ uniformly on all of X, and prove that $g \circ f_n \to g \circ f$ uniformly on all of X.

Let ρ_Y and ρ_Z be the metrics on Y and Z.

Set

$$T = \mathrm{Ran}(f) \cup \bigcup_{n=1}^{\infty} \mathrm{Ran}(f_n) \subset Y.$$

We claim that T is compact. To prove this, let $\mathcal{U}$ be an open cover of T. Since $\mathrm{Ran}(f)$ is compact, there are $k \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_k \in \mathcal{U}$ such that the open set $W = \bigcup_{j=1}^{k} U_j$ contains $\mathrm{Ran}(f)$. Set $\delta = \mathrm{dist}(\mathrm{Ran}(f), Y \setminus W)$. Then $\delta > 0$.

Choose $n \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ and all $x \in X$, we have $\rho_Y(f_n(x), f(x)) < \delta$. Then $\operatorname{Ran}(f_n) \subset W$ for all $n \geq N$. The set $T_0 = \bigcup_{n=1}^{N-1} \operatorname{Ran}(f_n)$ is compact. so there are $m \in \mathbb{Z}_{>0}$ and $V_1, V_2, \ldots, V_m \in \mathcal{U}$ which cover T_0. Then $U_1, U_2, \ldots, U_k, V_1, V_2, \ldots, V_m$ cover T_0. The claim is proved.

To prove the lemma, let $\varepsilon > 0$. Since T is compact, g is uniformly continuous on T. Therefore there is $\delta > 0$ such that whenever $y_1, y_2 \in T$ satisfy $\rho_Y(y_1, y_2) < \delta$, then $\rho_Z(g(y_1), g(y_2)) < \varepsilon$. Choose $n \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ and all $x \in X$, we have $\rho_Y(f_n(x), f(x)) < \delta$. Then also $\rho_Z\big((g \circ f_n)(x), \, (g \circ f)(x)\big) < \varepsilon$. This completes the proof. □

Lemma 8. Let $G \subset H(D)$ be

$$G = \big\{ h \in H(D) \colon h(z) < 1 \text{ for all } z \in D \text{ and } h(0) = 0 \big\}$$

Then G is a normal family. Moreover, if $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in G, and $f_n \to f$ uniformly on compact sets in D, then $f \in G$.

Proof. It follows from Theorem 14.6 of Rudin that G is a normal family.

For the second part, f is holomorphic by Theorem 10.28 of Rudin, and clearly $|f(z)| \leq 1$ for all $z \in D$ and $f(0) = 0$. The Schwarz Lemma (Theorem 12.2 of Rudin) implies that $|f(z)| < 1$ for all $z \in D$. Therefore $f \in G$. □

The following lemma solves the original problem, and is useful for the last question.

Lemma 9. Let $a \in P$. Then the set

$$F_a = \big\{ f \in H(D) \colon \operatorname{Re}(f(z)) > 0 \text{ for all } z \in D \text{ and } f(0) = a \big\}$$

is a normal family.

Proof. By the Riemann Mapping Theorem (Theorem 14.8 and Remark 14.9 of Rudin), there is a holomorphic function $g \colon P \to D$ which is bijective, has holomorphic inverse, and satisfies $g(a) = 0$.

Now let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in F_a. Then $(g \circ f_n)_{n \in \mathbb{Z}_{>0}}$ is a sequence in G. By Lemma 8, there exist a strictly increasing function $k \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ and a function $h \in H(D)$ such that $g \circ f_{k(n)} \to h$ uniformly on compact sets in D, and moreover $h \in G$. Therefore $g^{-1} \circ h$ is defined and is in $H(D)$. Lemma 7 implies that $f_n = g^{-1} \circ g \circ f_n \to g^{-1} \circ h$ uniformly on compact sets in D. This completes the proof that F_a is a normal family. □

Example 10. The condition "$f(0) = 1$" can't be omitted. Example: for $n \in \mathbb{Z}_{>0}$ define $f_n(z) = n$ for all $z \in D$. Then $\operatorname{Re}(f_n(z)) > 0$ for all $z \in D$ and $n \in \mathbb{Z}_{>0}$. However, $(f_n)_{n \in \mathbb{Z}_{>0}}$ has no subsequence which converges even pointwise to a function $f \colon D \to \mathbb{C}$.