MATH 618 (SPRING 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 7

This assignment is due on Canvas on Wednesday 22 May 2024 at 9:00 pm.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

The next problem counts as 1.5 ordinary problems.

Problem 1. Let $w_1, w_2, \ldots \in \mathbb{C} \setminus \{0\}$.

(1) Prove that if $\prod_{n=1}^{\infty} w_n$ converges to a nonzero value, then $\lim_{n \to \infty} w_n = 1$. Show that this can fail if $\prod_{n=1}^{\infty} w_n$ converges to 0.
(2) Prove that $\prod_{n=1}^{\infty} w_n$ converges to a nonzero value if and only if $\sum_{n=1}^{\infty} \log(w_n)$ converges.

The second part counts for about twice as much as the first.

I am sure this can be found in some textbook, but please work out the details yourself.

So as to ensure that $\sum_{n=1}^{\infty} \log(w_n)$ makes sense, take log to be defined on $\mathbb{C} \setminus \{0\}$ by $\log(re^{i\theta}) = \log(r) + i\theta$ when $r > 0$ and $\theta \in (-\pi, \pi]$, with $\log(r)$ using the usual definition of the logarithm as a function $(0, \infty) \to \mathbb{R}$.

There are two annoyances to deal with. First, we haven't formally proved that the definition $\log(re^{i\theta}) = \log(r) + i\theta$ gives a continuous function on $\mathbb{C} \setminus (-\infty, 0]$, since we never proved that $z \mapsto \arg(z)$ is continuous on any domain. (It is certainly not continuous on the domain given above.) You can prove this directly, but there are easier ways to proceed. You can use Problem 6 in Chapter 10 of Rudin's book (which was in a previous homework assignment), but this is overkill. Second, it is not generally true that $\log(ab) = \log(a) + \log(b)$, with any continuous definition on any nonempty neighborhood of 1 in $\mathbb{C}$.

The first solution for (1) is the direct solution, in terms of ε and δ. But the problem can be reduced to one about a ratio of sequences; see the alternate solution.

Solution for (1). Following the convention used in the lectures, set $p_n = \prod_{k=1}^{n} w_k$, and set $p = \lim_{n \to \infty} p_n$ when this limit exists.

For the first sentence, let $\varepsilon > 0$. Since $p \neq 0$, the number

$$\delta = \min\left(\frac{|p|}{2}, \frac{\varepsilon |p|}{4}\right)$$

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satisfies $\delta > 0$. Choose $N \in \mathbb{Z}_{>0}$ such that whenever $n \geq N$ then $|p_n - p| < \delta$. For $n \geq N+1$ we then have $|p_n| > \frac{1}{2}|p|$. Therefore

$$\begin{aligned} |w_n - 1| &= \left(\frac{1}{|p_n|}\right)|p_n - p_{n-1}| < \left(\frac{2}{|p|}\right)\varepsilon \\ &\leq \left(\frac{2}{|p|}\right)\big(|p_n - p| + |p_{n-1} - p|\big) < \left(\frac{2}{|p|}\right)\cdot 2\delta \leq \varepsilon. \end{aligned}$$

For the second sentence, take $w_n = \frac{1}{2}$ for all $n \in \mathbb{Z}_{>0}$. □

For the second sentence, taking $w_n = 0$ for all n does not work, since the problem statement says $w_n \in \mathbb{C} \setminus \{0\}$.

Alternate solution for (1). Following the convention used in the lectures, set $p_n = \prod_{k=1}^{n} w_k$, and set $p = \lim_{n\to\infty} p_n$ when this limit exists.

For the first sentence, observe that

$$w_n = \frac{p_n}{p_{n-1}}$$

when $n \geq 2$. Therefore

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} \frac{p_n}{p_{n-1}} = \frac{\lim_{n\to\infty} p_n}{\lim_{n\to\infty} p_{n-1}} = \frac{p}{p} = 1.$$

For the second sentence, take $w_n = \frac{1}{2}$ for all $n \in \mathbb{Z}_{>0}$. □

Solution for (2). We clearly have $\exp(\log(z)) = z$ for all $z \in \mathbb{C} \setminus \{0\}$.

First suppose $\sum_{n=1}^{\infty} \log(w_n)$ converges. For $n \in \mathbb{Z}_{>0}$, as in the solution to (1), set $p_n = \prod_{k=1}^{n} w_k$. Also set $s_n = \sum_{k=1}^{n} \log(w_k)$ and $s = \sum_{n=1}^{\infty} \log(w_n)$. Then $\exp(s_n) = p_n$. By continuity of exp, we get

$$\lim_{n\to\infty} p_n = \exp\left(\lim_{n\to\infty} s_n\right) = \exp(s).$$

Therefore $\prod_{n=1}^{\infty} w_n$ converges, and the product is nonzero since $\exp(s) \neq 0$.

For the reverse implication, first use $\exp'(0) \neq 0$ and the Inverse Function Theorem to choose $r > 0$, an open set $V \subset \mathbb{C}$ with $1 \in V$, and a bijection $h\colon V \to B_r(0)$ which is holomorphic and satisfies $h(\exp(z)) = z$ for all $z \in B_r(0)$ and $\exp(h(z)) = z$ for all $z \in V$. We may clearly require $r < \frac{\pi}{2}$ and

$$V \subset \big\{z \in \mathbb{C}\colon \mathrm{Re}(z) > 0\big\}. \tag{1}$$

We claim that, using the definition of log in the problem, $h(z) = \log(z)$ for all $z \in V$. To prove the claim, set $S = \big\{z \in \mathbb{C}\colon -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}\big\}$. For $z \in V$ we have $h(z) \in S$ because $r < \frac{\pi}{2}$, and $\log(z) \in S$ by (1). The restriction $\exp|_S$ is easily seen to be injective, and $\exp(h(z)) = z = \exp(\log(z))$. The claim follows.

Define $W = \big\{\exp(z)\colon z \in B_{r/2}(0)\big\}$. Then $1 \in W$, and W is open by the Open Mapping Theorem (or because $\exp\colon B_r(0) \to V$ is a homeomorphism).

We claim that if $a, b \in W$ then $ab \in V$ and $h(ab) = h(a) + h(b)$. We prove the claim. By construction, $h(a), h(b) \in B_{r/2}(0)$. Therefore $h(a) + h(b) \in B_r(0)$. So $ab = \exp(h(a))\exp(h(b)) = \exp(h(a) + h(b)) \in V$. This equation then implies that $h(ab) = h\big(\exp(h(a) + h(b))\big) = h(a) + h(b)$. The claim is proved.

Now suppose $\prod_{n=1}^{\infty} w_n$ converges to $p \neq 0$. Following the convention used in the lectures, set $p_n = \prod_{k=1}^{n} w_k$, so that $p = \lim_{n\to\infty} p_n$. It follows that $\lim_{n\to\infty} \frac{p}{p_n} = 1$. Therefore, also since $\exp(B_{r/4}(0))$ is open, there is $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$

we have $\frac{p}{p_n} \in \exp(B_{r/4}(0))$. By part (1), we can also require that N be so large that whenever $n \geq N$ we have $w_n \in W$.

Since $x, y \in B_{r/4}(0)$ implies $x - y \in B_{r/2}(0)$, it follows that if $a, b \in \exp(B_{r/4}(0))$ then $ab^{-1} \in W$. Therefore, for $m, n \geq N$, we have $\frac{p_m}{p_n} = \left(\frac{p}{p_n}\right)\left(\frac{p}{p_m}\right)^{-1} \in W$.

We claim that for all $n \geq N$ we have $h(p_n/p_N) = \sum_{k=N+1}^{n} h(w_k)$. The proof of the claim is by induction. It is trivially true for $n = N$, because $h(1) = 0$. Suppose it holds for n. Then

$$\frac{p_{n+1}}{p_N} = \left(\frac{p_n}{p_N}\right) w_{n+1}.$$

Both factors on the right are in W. Using this at the first step, and the induction hypothesis at the second step, we get

$$h\left(\frac{p_{n+1}}{p_N}\right) = h\left(\frac{p_n}{p_N}\right) + h(w_{n+1}) = \sum_{k=N+1}^{n+1} h(w_k),$$

completing the induction and the proof of the claim.

Since $\prod_{n=1}^{\infty} w_n$ converges, $\lim_{n\to\infty} \frac{p_n}{p_N}$ exists. Call it λ. Then $\lambda \in \overline{W} \subset V$. By continuity of h, and since $w_k \in W \subset V$ for $k \geq N$, we have

$$\sum_{k=N+1}^{\infty} \log(w_k) = \sum_{k=N+1}^{\infty} h(w_k) = \lim_{n\to\infty} \sum_{k=N+1}^{n} h(w_k) = h(\lambda).$$

So $\sum_{k=N+1}^{\infty} \log(w_k)$ converges. Therefore $\sum_{k=1}^{\infty} \log(w_k)$ converges, as desired. $\square$

Problem 2 (Problem 19 in Chapter 14 of Rudin's book, plus an additional statement)**.** Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disk. If $f \colon D \to D$ is holomorphic and bijective, prove that f extends to a homeomorphism from $\overline{D}$ to $\overline{D}$. (This part is essentially immediate.) Then (this is the main part) exhibit, of course with proof, a bijective homeomorphism $f \colon D \to D$ which does not extend to a continuous function from $\overline{D}$ to $\overline{D}$.

The first part is included mainly for context.

Solution. For the first statement, use Theorem 12.6 of Rudin (proved in class) to find $\alpha \in D$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that for all $z \in D$, we have

$$f(z) = \lambda\left(\frac{z - \alpha}{1 - \overline{\alpha} z}\right).$$

By Theorem 12.4 of Rudin (proved in class), this function extends to a bijective holomorphic function $\varphi_\alpha \colon \mathbb{C} \setminus \{1/\overline{\alpha}\} \to \mathbb{C} \setminus \{-1/\overline{\alpha}\}$ with $\varphi_\alpha(\partial D) = \partial D$, and in particular to a homeomorphism $\overline{D} \to \overline{D}$.

For the second part, define $f \colon D \to D$ by

$$f(z) = \exp\left(\frac{i}{1 - |z|}\right) z.$$

(In polar coordinates,

$$f(re^{i\theta}) = r \exp\left(i\left(\theta + \frac{1}{1 - r}\right)\right)$$

when $r \in [0, 1)$ and $\theta \in \mathbb{R}$.)

This function clearly has the continuous inverse

$$z \mapsto \exp\left(-\frac{i}{1-|z|}\right) z.$$

Therefore f is a homeomorphism.

To prove that f does not extend to a continuous function from $\overline{D}$ to $\overline{D}$, it suffices to find a sequence $(z_n)_{n\in\mathbb{Z}_{>0}}$ in D such that $\lim_{n\to\infty} z_n$ exists but $\lim_{n\to\infty} f(z_n)$ does not exist. Take

$$z_n = 1 - \frac{1}{\pi n}.$$

Then $z_n \in D$, $\lim_{n\to\infty} z_n = 1$, but $f(z_n) = (-1)^n z_n$, so $(f(z_n))_{n\in\mathbb{Z}_{>0}}$ has two cluster points, namely 1 and -1. □

Problem 3 (Problem 22 in Chapter 14 of Rudin's book)**.** Let $D = \{z \in \mathbb{C}\colon |z| < 1\}$ be the open unit disk. Let Ω be an open square with center at 0 (but with sides not necessarily parallel to the coordinate axes), and let $f\colon D \to \Omega$ is holomorphic, bijective, and satisfy $f(0) = 0$. Prove that $f(iz) = if(z)$ for all $z \in D$. If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, prove that $c_n = 0$ unless $n-1$ is a multiple of 4. Generalize by replacing squares with general rotationally symmetric simply connected regions (of course other than $\mathbb{C}$).

We first state a lemma.

Lemma 1. Let $\Omega \subset \mathbb{C}$ be a simply connected region with $\Omega \neq \mathbb{C}$, let $w_0 \in \Omega$, and let $f, g\colon D \to \Omega$ be holomorphic, bijective, and satisfy $f(0) = g(0) = w_0$. Then there is $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $g(z) = f(\lambda z)$ for all $z \in D$.

Proof. Set $h = f^{-1} \circ g$. Then $h\colon D \to D$ is holomorphic, bijective, and satisfies $f(0) = 0$. Theorem 12.6 of Rudin (proved in class) provides $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $h(z) = \lambda z$ for all $z \in D$. Apply f. □

Here is the general statement.

Theorem 2. Let $\Omega \subset \mathbb{C}$ be a simply connected region such that $0 \in \Omega$. Let $\omega \in \mathbb{C}$ satisfy $|\omega| = 1$ and $\omega\Omega = \Omega$. Let $f\colon D \to \Omega$ is holomorphic, bijective, and satisfy $f(0) = 0$. Then:

(1) $f(\omega z) = \omega f(z)$ for all $z \in D$.
(2) If we write $f(z) = \sum_{n=0}^{\infty} c_n z^n$, and $m \in \mathbb{Z}_{\geq 0}$ is a number such that $c_m \neq 0$, then $\omega^{m-1} = 1$.

Proof. For the first part, observe that $g(z) = \omega f(z)$ is also a holomorphic bijection from D t Ω. By Lemma 1, there is $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $g(z) = f(\lambda z)$ for all $z \in D$. Moreover, using the Chain Rule for the second step, $\omega f'(0) = g'(0) = \lambda f'(0)$. We have $f'(0) \neq 0$ because f is injective, so $\lambda = \omega$.

For the second part, for $z \in D$ we have

$$\sum_{n=0}^{\infty} c_n z^n = f(z) = \omega^{-1} f(\omega z) = \omega^{-1} \sum_{n=0}^{\infty} c_n (\omega z)^n = \sum_{n=0}^{\infty} \omega^{n-1} c_n z^n.$$

By uniqueness of power series, $(\omega^{n-1} - 1)c_n = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Part (2) follows. □

Solution to the original statement. If Ω is a square centered at 0, then $i\Omega = \Omega$. Take $\omega = i$ in Theorem 2. Part (1) gives $f(iz) = if(z)$ for all $z \in D$, and, since $i^m = 1$ if and only if m is divisible by 4, part (2) gives $c_n = 0$ unless $n - 1$ is a multiple of 4. $\square$

The next problem counts as 1.5 ordinary problems.

Problem 4 (Problem 29 in Chapter 14 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a bounded region, fix $a \in \Omega$, and let $f\colon \Omega \to \Omega$ be a holomorphic function such that $f(a) = a$.

(1) Prove that $|f'(a)| \leq 1$. Hint: set $f_1 = f$ and inductively define $f_{n+1} = f \circ f_n$. (These functions are normally called f^n.) Compute $f_n'(a)$.
(2) If $f'(a) = 1$, prove that $f(z) = z$ for all $z \in \Omega$. Hint: if $f(z) = z + c_m(z - a)^m + c_{m+1}(z - a)^{m+1} + \cdots$, compute the coefficient of $(z - a)^m$ in the expansion of $f_n(z)$.
(3) If $|f'(a)| = 1$, prove that f is bijective. Hint: set $\gamma = f'(a)$. Find positive integers $k(1) < k(2) < \cdots$ such that $\lim_{n\to\infty} \gamma^{k(n)} = 1$ and the sequence $(f_{k(n)})_{n\in\mathbb{Z}_{>0}}$ converges uniformly on compacts sets in Ω to some function g. Prove that $g'(a) = 1$. Use Problem 20 in Chapter 10 of Rudin's book (in a previous homework assignment) to prove that $g(\Omega) \subset \Omega$. Use these facts to deduce the desired conclusions for f.

Remark 3. The functions f_n in the hint for part (1) should properly be called f^n. This is consistent with the notation f^{-1} for the inverse function. (So, if $f\colon X \to X$ is invertible then f^{-2} makes sense.) It is standard in work on dynamical systems. For this reason, notation like "$\sin^2(x)$" is bad.

One major exception (another example of there being not enough notation to go around): if X is, say, a compact Hausdorff space, then the set $C(X)$ of continuous functions from X to $\mathbb{C}$ is an algebra, and in particular a ring. With respect to the ring operations, the function $x \mapsto [f(x)]^n$ must be called f^n, and f^{-1} is the function $f^{-1}(x) = 1/f(x)$.

If you are a C*-algebraist and work on crossed products by minimal homeomorphisms, then both meanings may occur in the same sentence.

We collect for reference some repeatedly used easy facts.

Notation 4. Throughout, $\Omega \subset \mathbb{C}$ be a bounded region, $a \in \Omega$, and $f\colon \Omega \to \Omega$ is a holomorphic function such that $f(a) = a$. Moreover, the functions $f_n\colon \Omega \to \Omega$ are defined inductively by $f_1 = f$ and $f_{n+1} = f \circ f_n$ for $n \in \mathbb{Z}_{>0}$. The set F is defined to be the set of all holomorphic functions from Ω to Ω.

Lemma 5. Adopt Notation 4. Then $f_n'(a) = f'(a)^n$ for all $n \in \mathbb{Z}_{>0}$.

Proof. This is an immediate induction argument using the Chain Rule. $\square$

Lemma 6. Adopt Notation 4. For every sequence $(h_n)_{n\in\mathbb{Z}_{>0}}$ in F, there are a subsequence $(h_{k(n)})_{n\in\mathbb{Z}_{>0}}$ of $(h_n)_{n\in\mathbb{Z}_{>0}}$ and a function $g\colon \Omega \to \mathbb{C}$ such that, for all $m \in \mathbb{Z}_{\geq 0}$, $h_{k(n)}^{(m)} \to g^{(m)}$ uniformly on compact subsets of Ω.

Proof. The set F is uniformly bounded because Ω is bounded. So F is a normal family by Theorem 14.6 of Rudin. Thus, there are a subsequence $(h_{k(n)})_{n\in\mathbb{Z}_{>0}}$ of $(h_n)_{n\in\mathbb{Z}_{>0}}$ and a function $g\colon \Omega \to \mathbb{C}$ such that $h_{k(n)} \to g$ uniformly on compact subsets of Ω. By applying Theorem 10.28 of Rudin m times, g is holomorphic and $f_{k(n)}^{(m)} \to g^{(m)}$ uniformly on compact subsets of Ω. $\square$

Lemma 7. Adopt Notation 4. There are positive integers $k(1) < k(2) < \cdots$ such that $\lim_{n\to\infty} f'(a)^{k(n)}$ exists.

Proof. Apply Lemma 6 to get a subsequence $(f_{k(n)})_{n\in\mathbb{Z}_{>0}}$ of $(f_n)_{n\in\mathbb{Z}_{>0}}$ and a function $g\colon \Omega \to \mathbb{C}$ such that $f'_{k(n)} \to g'$ uniformly on compact subsets of Ω. Then Lemma 5 says $\lim_{n\to\infty} f'(a)^{k(n)} = g'(a)$. $\square$

Solution to part (1). Adopt Notation 4. Lemma 7 gives integers $k(n) \to \infty$ such that $\lim_{n\to\infty} f'(a)^{k(n)}$ exists. It is immediate that $|f'(a)| \leq 1$. $\square$

The following solution doesn't depend on normal families.

Alternate solution to part (1). Adopt Notation 4. Choose $r > 0$ such that $\overline{B_r(a)} \subset \Omega$ and $R > 0$ such that $\Omega \subset B_R(a)$. For $n \in \mathbb{Z}_{>0}$, use $f_n(\Omega) \subset \Omega$ and Cauchy's Estimates (see Theorem 10.22 of Rudin's book) to get

$$|f_n'(a)|^2 r^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| f(a + re^{i\theta}) \right|^2 d\theta \leq R^2.$$

Combining this with Lemma 5 gives

$$|f'(a)|^n = |f_n'(a)| \leq \frac{R}{r}.$$

Since the right hand side is independent of n, it follows that $|f'(a)| \leq 1$. $\square$

For part (2), we use the hint in a slightly different form than was suggested. (If one takes it literally, one needs to address convergence issues. This isn't hard but is annoying.) it is convenient to separate the following calculation.

Lemma 8. Let $\Omega \subset \mathbb{C}$ be open, let h be a holomorphic function on Ω, let $a \in \Omega$, and let $m \in \mathbb{Z}_{>0} \setminus \{1\}$. Define $f(z) = z + (z-a)^m h(z)$. Then $f^{(m)}(a) = m!h(a)$.

Proof. First, induction and the product rule show that if g and k are holomorphic functions on Ω and $n \in \mathbb{Z}_{\geq 0}$, then

$$(g \cdot k)^{(n)} = \sum_{l=0}^{n} \binom{n}{l} g^{(l)} \cdot k^{(n-l)}.$$

Applying this formula to f, since $m \geq 2$ we get

$$f^{(m)}(z) = \sum_{l=0}^{m} \binom{m}{l} m(m-1)\cdots(m-l+1)(z-a)^{m-l} \cdot h^{(m-l)}(z).$$

If we put $z = a$, the only nonzero term is the one for $l = m$. $\square$

Solution to part (2). Suppose the statement is false. Adopt Notation 4.

Let m be the least integer $m \geq 2$ such that the coefficient c_m in the power series expansion $f(z) = \sum_{n=0}^{\infty} c_m (z-a)^n$ is nonzero. (That is, $m-2$ is the order of the zero of f'' at a; this order might be zero.) The hypotheses imply that $c_0 = a$ and $c_1 = 1$. So there is a holomorphic function g on Ω such that $g(a) \neq 0$ and for all $z \in \Omega$ we have

$$f(z) = z + (z-a)^m g(z). \tag{2}$$

Inductively define holomorphic functions g_n on Ω by $g_1 = g$ and

$$g_{n+1}(z) = g_n(z) + [1 + (z-a)^{m-1} g_n(z)]^m g(f_n(z))$$

for $n \in \mathbb{Z}_{>0}$ and $z \in \Omega$. We claim that for $n \in \mathbb{Z}_{>0}$ we have $f_n(z) = z + (z - a)^m g_n(z)$. We prove the claim by induction. It is true for $n = 1$ by (2). Suppose it holds for n. By (2) and the induction hypothesis, we have

$$\begin{aligned} f_{n+1}(z) = f(f_n(z)) &= f_n(z) + (f_n(z) - a)^m g(f_n(z)) \\ &= z + (z-a)^m g_n(z) + \left[z + (z-a)^m g_n(z) - a\right]^m g(f_n(z)) \\ &= z + (z-a)^m \left[g_n(z) + [1 + (z-a)^{m-1} g_n(z)]^m g(f_n(z))\right] \\ &= z + (z-a)^m g_{n+1}(z), \end{aligned}$$

as desired. The claim is proved.

Since $f_n(a) = a$ and $m \geq 2$, an induction argument now shows that $g_n(a) = ng(a)$ for all $n \in \mathbb{Z}_{>0}$. So Lemma 8 yields $f_n^{(m)}(a) = nm!g(a)$ for all $n \in \mathbb{Z}_{>0}$.

By Lemma 6, there are a subsequence $(f_{k(n)})_{n \in \mathbb{Z}_{>0}}$ of $(f_n)_{n \in \mathbb{Z}_{>0}}$ and a holomorphic function $g \colon \Omega \to \mathbb{C}$ such that $\lim_{n\to\infty} f_{k(n)}^{(m)}(a) = g^{(m)}(a)$. But, since $g(a) \neq 0$, the previous paragraph shows that $\lim_{n\to\infty} f_{k(n)}^{(m)}(a) = \infty$. This contradiction completes the solution. $\square$

Alternate solution to part (2). Argue as in the first solution to part (2) through all but the last paragraphs, to deduce that, if the conclusion is false, there are $m \geq 2$ and $b \in \mathbb{C} \setminus \{0\}$ (equal to $g'(a)$ there) such that

$$f_n^{(m)}(a) = nm!b \tag{3}$$

for all $n \in \mathbb{Z}_{>0}$. Choose $r > 0$ such that $\overline{B_r(a)} \subset \Omega$ and $R > 0$ such that $\Omega \subset B_R(a)$. For $n \in \mathbb{Z}_{>0}$, use $f_n(\Omega) \subset \Omega$ and Cauchy's Estimates (see Theorem 10.22 of Rudin's book) to get

$$\left|\frac{f_n^{(m)}(a)}{m!}\right|^2 r^{2m} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(a + re^{i\theta})\right|^2 d\theta \leq R^2.$$

Combining this with 3 gives

$$n|b|^m = \left|\frac{f_n^{(m)}(a)}{m!}\right| \leq \frac{R}{r^m}.$$

Since the right hand side is independent of n, this shows that $|b|^m = 0$, a contradiction. $\square$

Solution to part (3). Adopt Notation 4.

Set $\gamma = f'(a)$. Apply Lemma 7, getting positive integers $l_0(1) < l_0(2) < \cdots$ snd $\lambda \in \mathbb{C}$ such that $\lim_{n\to\infty} \gamma^{l_0(n)} = \lambda$. Passing to a subsequence, find positive integers $l(1) < l(2) < \cdots$ such that $l(n+1) > 2l(n)$ for all $n \in \mathbb{Z}_{>0}$ and $\lim_{n\to\infty} \gamma^{l(n)} = \lambda$. For $n \in \mathbb{Z}_{>0}$, set $k_0(n) = l(n+1) - l(n)$. Then $k_0(1) < k_0(2) < \cdots$ and

$$\lim_{n\to\infty} \gamma^{k_0(n)} = \lim_{n\to\infty} \gamma^{l(n+1)} \left(\gamma^{l(n)}\right)^{-1} = 1.$$

By Lemma 6, there are a further subsequence $(f_{k(n)})_{n \in \mathbb{Z}_{>0}}$ of $(f_{k_0(n)})_{n \in \mathbb{Z}_{>0}}$ and a holomorphic function $g \colon \Omega \to \mathbb{C}$ such that $f_{k(n)} \to g$ uniformly on compact subsets of Ω, and also $\lim_{n\to\infty} f'_{k(n)}(a) = g'(a)$. Lemma 5 gives $g'(a) = \lim_{n\to\infty} \gamma^{k_0(n)} = 1$. So g is not constant. Problem 20 in Chapter 10 of Rudin's book now implies that $g(\Omega) \subset \Omega$. Part (2) implies that $g(z) = z$ for all $z \in \Omega$.

We claim that f is injective. Suppose $f(z_1) = f(z_2)$. Then $f_n(z_1) = f_n(z_2)$ for all $n \in \mathbb{Z}_{>0}$, so

$$z_1 = g(z_1) = \lim_{n\to\infty} f_{k(n)}(z_1) = \lim_{n\to\infty} f_{k(n)}(z_2) = g(z_2) = z_2.$$

The claim is proved.

It remains to prove that f is surjective. Let $w \in \Omega$. Choose $r > 0$ such that $\overline{B_r(w)} \subset \Omega$. Choose $n \in \mathbb{Z}_{>0}$ with $k(n) > 1$ such that

$$\sup_{z\in\overline{B_r(w)}} |f_{k(n)}(z) - g(z)| < \frac{r}{2}.$$

Since $|g(z)-w| = r$ for $z \in \partial B_r(w)$, Rouché's Theorem says that $g-w$ and $f_{k(n)}-w$ have the same number of zeros in $B_r(w)$. Since $g - w$ vanishes at w, it follows that there is $z \in \Omega$ such that $f_{k(n)}(z) = w$. That is, $w = f_{k(n)}(z) = f(f_{k(n)-1}(z)) \in \operatorname{Ran}(f)$. $\square$