MATH 618 (SPRING 2024, PHILLIPS): HOMEWORK 8

This assignment is due on Canvas on Monday 3 June 2024 at 9:00 pm.

Problems from Chapters 12 and 13 of Rudin do not, I hope, depend on material from those chapters not discussed in the course.

Problem 1 (Problem 18 in Chapter 14 of Rudin's book)**.** Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disk. Let Ω be a simply connected region with $\Omega \neq \mathbb{C}$, and let $z_0 \in \Omega$. Suppose $f, g \colon \Omega \to D$ are holomorphic and bijective, and satisfy $f(z_0) = g(z_0) = 0$. How are f and g related? Now let $a \in D$ be arbitrary. If $f, g \colon \Omega \to D$ are holomorphic and bijective, and satisfy $f(z_0) = g(z_0) = a$, how are f and g related?

Problem 2 (Problem 3 in Chapter 12 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region. Determine exactly when a holomorphic function f has the property that $|f|$ has a local *minimum* on Ω.

Problem 3 (Problem 4 in Chapter 12 of Rudin's book)**.**

(1) Let $\Omega \subset \mathbb{C}$ be a region, let $a \in \Omega$, and let $r > 0$ satisfy $\overline{B_r(a)} \subset \Omega$. Let f be a nonconstant holomorphic function on Ω such that $|f|$ is constant on $\partial B_r(a)$. Prove that f has a zero in $B_r(a)$.

(2) Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$.

Problem 4 (Problem 5 in Chapter 12 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a bounded region, let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $C(\overline{\Omega})$ such that $f_n|_{\Omega}$ is holomorphic for all $n \in \mathbb{Z}_{>0}$, and suppose that there is a function g on $\partial\Omega$ such that $(f_n|_{\partial\Omega})_{n \in \mathbb{Z}_{>0}}$ converges uniformly to g. Prove that there is a function f on Ω such that $(f_n)_{n \in \mathbb{Z}_{>0}}$ converges uniformly to f.

Problem 5 (Problem 10 in Chapter 13 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, and let f be a holomorphic function on Ω which is not the constant function zero. Suppose that for every $n \in \mathbb{Z}_{>0}$ there is a holomorphic function g on Ω such that $g(z)^n = f(z)$ for every $z \in \Omega$. Prove that there is a holomorphic function h on Ω such that $\exp(h(z)) = f(z)$ for every $z \in \Omega$.

Date: 22 May 2024.