MATH 618 (SPRING 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 8

This assignment is due on Canvas on Monday 3 June 2024 at 9:00 pm.

Problems from Chapters 12 and 13 of Rudin do not, I hope, depend on material from those chapters not discussed in the course.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Problem 1 (Problem 18 in Chapter 14 of Rudin's book)**.** Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disk. Let Ω be a simply connected region with $\Omega \neq \mathbb{C}$, and let $z_0 \in \Omega$. Suppose $f, g \colon \Omega \to D$ are holomorphic and bijective, and satisfy $f(z_0) = g(z_0) = 0$. How are f and g related? Now let $a \in D$ be arbitrary. If $f, g \colon \Omega \to D$ are holomorphic and bijective, and satisfy $f(z_0) = g(z_0) = a$, how are f and g related?

Lemma 1.

Proof. □

Solution. □

Alternate solution (sketch). □

Problem 2 (Problem 3 in Chapter 12 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region. Determine exactly when a holomorphic function f has the property that $|f|$ has a local *minimum* on Ω.

Solution. We claim that this happens if and only if f is constant or there exists $z \in \Omega$ such that $f(z) = 0$.

First, if one or other of these conditions is satisfied, then obviously $|f|$ has a local minimum on Ω.

For the reverse, assume that f vanishes nowhere on Ω and that $|f|$ has a local minimum on Ω: there is $a \in \Omega$ such that $|f(z)| \geq |f(a)|$ for all $z \in \Omega$.. Since f vanishes nowhere on Ω, the function $z \mapsto g(z) = f(z)^{-1}$ is holomorphic on Ω. Clearly g has a local maximum at a. Therefore g is constant by the Maximum Modulus Theorem. So f is constant. □

Problem 3 (Problem 4 in Chapter 12 of Rudin's book)**.**

(1) Let $\Omega \subset \mathbb{C}$ be a region, let $a \in \Omega$, and let $r > 0$ satisfy $\overline{B_r(a)} \subset \Omega$. Let f be a nonconstant holomorphic function on Ω such that $|f|$ is constant on $\partial B_r(a)$. Prove that f has a zero in $B_r(a)$.

(2) Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$.

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Solution for part (1). Suppose $|f|$ is constant on $\partial B_r(a)$ and $f(z) \neq 0$ for all $z \in B_r(a)$. Let M be the constant value of $|f|$ on $\partial B_r(a)$. By compactness, $|f|$ has a global maximum on $\overline{B_r(a)}$, say at $z \in \overline{B_r(a)}$. If $|f(z)| > M$, then $z \notin \partial B_r(a)$, so $|f|$ has a local maximum at $z \in B_r(a)$. The function f is not constant because $|f(z)| > M$, so this contradicts the Maximum Modulus Theorem. Therefore $|f| \leq M$ on $B_r(a)$. Since $f(z) \neq 0$ for all $z \in B_r(a)$, it follows that $M > 0$.

Similarly, and using $M > 0$, the function $z \mapsto g(z) = f(z)^{-1}$ is holomorphic on an open subset $U \subset \Omega$ and satisfies $|g| \leq M^{-1}$ on $B_r(a)$. Combining this with the previous inequality, we get $|f| = M$ on $B_r(a)$. So f is constant by the Maximum Modulus Theorem. □

Solution for part (2) (sketch). Set $D = \{z \in \mathbb{C} \colon |z| < 1\}$. Recall the functions

$$\varphi_a(z) = \frac{z - a}{1 - \overline{a}z}$$

for $a \in D$, each defined on a neighborhood of $\overline{D}$.

Let $a_1, a_2, \ldots, a_n \in D$ be the zeros of f in D, with multiplicities $m_1, m_2, \ldots, m_n \in \mathbb{Z}_{>0}$. There is a holomorphic function g defined on some neighborhood U of $\overline{D}$ such that

$$f(z) = \varphi_{a_1}(z)^{m_1} \cdot \varphi_{a_2}(z)^{m_2} \cdot \cdots \cdot \varphi_{a_n}(z)^{m_n},$$

and g is never zero on D and satisfies $|g(z)| = 1$ for all $z \in \partial D$ (because this is true of f and $\varphi_{a_1}, \varphi_{a_2}, \ldots, \varphi_{a_n}$). Therefore g is constant by part (1), and the constant λ must satisfy $|\lambda| = 1$. Now check which of the functions

$$z \mapsto \lambda \varphi_{a_1}(z)^{m_1} \cdot \varphi_{a_2}(z)^{m_2} \cdot \cdots \cdot \varphi_{a_n}(z)^{m_n}$$

extends to an entire function. □

Problem 4 (Problem 5 in Chapter 12 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a bounded region, let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $C\big(\overline{\Omega}\big)$ such that $f_n|_{\Omega}$ is holomorphic for all $n \in \mathbb{Z}_{>0}$, and suppose that there is a function g on $\partial\Omega$ such that $(f_n|_{\partial\Omega})_{n \in \mathbb{Z}_{>0}}$ converges uniformly to g. Prove that there is a function f on Ω such that $(f_n)_{n \in \mathbb{Z}_{>0}}$ converges uniformly to f.

Solution. It is enough to prove that $(f_n)_{n \in \mathbb{Z}_{>0}}$ is uniformly Cauchy. So let $\varepsilon > 0$.

Since $\overline{\Omega}$ is compact, for every $m, n \in \mathbb{Z}_{>0}$ there is $a_{m,n} \in \overline{\Omega}$ such that $|f_m(z) - f_n(z)| \leq |f_m(a_{m,n}) - f_n(a_{m,n})|$ for all $z \in \overline{\Omega}$. By the Maximum Modulus Theorem, we may choose $a_{m,n} \in \partial\Omega$. Since the sequence $(f_n|_{\partial\Omega})_{n \in \mathbb{Z}_{>0}}$ converges uniformly, it is uniformly Cauchy. Therefore there is $N \in \mathbb{Z}_{>0}$ such that whenever $m, n \geq N$ then

$$\sup_{z \in \partial\Omega} |f_m(z) - f_n(z)| < \varepsilon.$$

In particular, $|f_m(a_{m,n}) - f_n(a_{m,n})| < \varepsilon$. By the choice of $a_{m,n}$, we have

$$\sup_{z \in \partial\Omega} |f_m(z) - f_n(z)| \leq |f_m(a_{m,n}) - f_n(a_{m,n})| < \varepsilon.$$

So $(f_n)_{n \in \mathbb{Z}_{>0}}$ is uniformly Cauchy, as desired. □

Problem 5 (Problem 10 in Chapter 13 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, and let f be a holomorphic function on Ω which is not the constant function zero. Suppose that for every $n \in \mathbb{Z}_{>0}$ there is a holomorphic function g on Ω such that $g(z)^n = f(z)$ for every $z \in \Omega$. Prove that there is a holomorphic function h on Ω such that $\exp(h(z)) = f(z)$ for every $z \in \Omega$.

Solution (sketch). We first claim that $f(z) \neq 0$ for all $z \in \Omega$. Otherwise, a zero of f must be isolated. If $g^n = f$, then g also has an isolated zero at z. The theorem on local behavior of g near a zero can be used to show that the order of the zero of f at z must be divisible by n. This can't be true for all $n \in \mathbb{Z}_{>0}$, proving the claim.

Next, we claim that if $\gamma\colon [\alpha, \beta] \to \Omega$ is any piecewise C^1 closed curve, then

$$\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = 0.$$

For any holomorphic function g on Ω which does not vanish on $\operatorname{Ran}(\gamma)$, consider the function

$$\varphi(t) = \int_\alpha^t \left(\frac{g'(\gamma(t))}{g'(\gamma(t))} \right) \gamma'(t)\, dt.$$

The function

$$\psi(t) = \frac{\exp(\varphi(t))}{g(\gamma(t))}$$

has $\psi'(t) = 0$ (except at finitely many points), so is constant. This shows that

$$\int_\gamma \frac{g'(\zeta)}{g(\zeta)}\, d\zeta \in 2\pi i \mathbb{Z}.$$

Now let $n \in \mathbb{Z}_{>0}$ be arbitrary, and choose g such that $g^n = f$. Then

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{n g'(\zeta)}{g(\zeta)} \qquad \text{and} \qquad \int_\gamma \frac{g'(\zeta)}{g(\zeta)}\, d\zeta \in 2\pi i \mathbb{Z},$$

so

$$\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\, d\zeta \in 2\pi i n \mathbb{Z}.$$

Since n is arbitrary, the claim follows.

Using the claim, show by methods we have already seen that there is a function h_0 on Ω such that

$$h_0'(z) = \frac{f'(z)}{f(z)}$$

for all $z \in \Omega$. Use a suitable constant multiple of h_0 to solve the problem. $\square$