MATH 618 (SPRING 2024, PHILLIPS): HOMEWORK 9

This assignment is due on Canvas on Monday 10 June 2024 at 9:00 pm.

Problem 1 (Problem 9a in Chapter 14 of Rudin's book)**.** Set

$$\Omega = \big\{ z \in \mathbb{C} \colon -1 < \mathrm{Re}(z) < 1 \big\} \qquad \text{and} \qquad D = \big\{ z \in \mathbb{C} \colon |z| < 1 \big\}.$$

Find an explicit formula for the conformal bijection $f \colon \Omega \to D$ such that $f(0) = 0$ and $f'(0) > 0$. Find $f'(0)$ for this function f.

Problem 2 (Problem 11 in Chapter 14 of Rudin's book)**.** Set

$$D = \big\{ z \in \mathbb{C} \colon |z| < 1 \big\} \qquad \text{and} \qquad \Omega = \big\{ z \in D \colon \mathrm{Im}(z) > 0 \big\}.$$

Find an explicit formula for the conformal bijection $f \colon \Omega \to D$ which has a continuous extension to a function $g \colon \overline{\Omega} \to \overline{D}$ satisfying $g(-1) = -1$, $g(0) = i$, and $g(1) = 1$. Find $z \in \Omega$ such that $f(z) = 0$. Find $f(i/2)$.

Hint: $f = \varphi \circ s \circ \psi$ for fractional linear transformations φ and ψ, and with $s(z) = z^2$ for $z \in \mathbb{C}$.

The next problem counts as two ordinary problems. (I don't yet know how hard it actually is.)

Problem 3. Iterate a suitable modification of the method presented in class to extend the Riemann zeta function over $\{ z \in \mathbb{C} \colon \mathrm{Re}(z) > 0 \} \setminus \{1\}$ to show that the Riemann zeta function extends to a holomorphic function on $\mathbb{C} \setminus \{1\}$.

Recall the method. We showed that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s} \, ds = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) ds,$$

and that the series in the last expression converges uniformly on compact sets in $\{ z \in \mathbb{C} \colon \mathrm{Re}(z) > 0 \}$ to a holomorphic function defined on this set.

Date: 3 June 2024.