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MATH 617 (WINTER 2024, PHILLIPS): EXERCISES (VERSION 1)

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Not enough proofreading has been done.

The following conventions hold unless explicitly stated otherwise. All topological groups are assumed Hausdorff. All homomorphisms between C*-algebras, including automorphisms, are assumed to be *-homomorphisms. All group actions are assumed continuous, and group actions on C*-algebras are assumed to be via *-automorphisms. Unitary representations are assumed to be continuous. If G is a group, then $L^p(G)$ is taken with respect to some (implicit) left Haar measure on G. A Haar measure μ on a compact group G is taken to satisfy $\mu(G) = 1$. Haar measure on a discrete group is taken to be counting measure.

Problem 1. Let G be a second countable locally compact group, with left Haar measure μ. For $g \in G$ define $w_g \colon L^2(G, \mu) \to L^2(G, \mu)$ by $(w_g \xi)(h) = \xi(g^{-1}h)$ for $\xi \in L^2(G, \mu)$ and $g, h \in G$. Show that w_g is in $L(L^2(G, \mu))$, that w_g is unitary, and that $g \mapsto w_g$ is a unitary representation.

The main point is to verify the appropriate continuity condition.

Second countability is not needed, but one must be more careful with the measure theory. One needs to know that $C_{\mathrm{c}}(G)$ is dense in $L^2(G)$.

Problem 2. Let G be a topological group. Give careful definitions of the following concepts.

(1) Unitary equivalence of unitary representations.
(2) Invariant subspace of a unitary representation.
(3) Irreducible unitary representation.
(4) Direct sum of an arbitrary family of unitary representations.
(5) Cyclic vector for a unitary representation. (Caution: if f w is a unitary representation of G on a Hilbert space H, and $\xi \in H$, then $\{w_g \xi \colon g \in G\}$ is usually not a subspace.)

Then give direct proofs of the following facts.

(6) If w is a unitary representation of G on a Hilbert space H, and $M \subset H$ is an invariant subspace for w (closed or not), then $M^{\perp}$ is a closed invariant subspace for w.
(7) The direct sum of an arbitrary family of unitary representations is again a unitary representation. (The main issue is continuity.)
(8) Every unitary representation of G is unitarily equivalent to a direct sum of cyclic unitary representations.
(9) Let w be a unitary representation of G on a Hilbert space H. Then
$$\big\{ a \in L(H) \colon a w_g = w_g a \text{ for all } g \in G \big\}$$
is a von Neumann algebra.

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(10) A unitary representation w of G on a Hilbert space H is irreducible if and only if
$$\big\{a \in L(H) \colon a w_g = w_g a \text{ for all } g \in G\big\} = \mathbb{C} \cdot 1_H.$$

As stated, these don't directly reduce to the analogous concepts and statements for C*-algebras unless the group is locally compact. Prove, however, that the definitions in (1), (2), (3), (4), and (5) are unchanged if the topology on the group is replaced with the discrete topology.

Problem 3. Let G be a group with the discrete topology. Prove that $\mathbb{C}[G]$ is a unital complex *-algebra.

Problem 4. Let G be a second countable locally compact group, with left Haar measure μ. For $a, b \in C_{\mathrm{c}}(G)$, define

$$(a * b)(g) = \int_G a(h) b(h^{-1} g) \, d\mu(h). \tag{1}$$

For $a \in C_{\mathrm{c}}(G)$, define, using the modular function Δ of G in the first,

$$a^*(g) = \Delta(g)^{-1} \overline{a(G^{-1})} \tag{2}$$

for $g \in G$ and

$$\|a\|_1 = \int_G |a(g)| \, d\mu(g). \tag{3}$$

Prove the following:

(1) In (1), if $a, b \in C_{\mathrm{c}}(G)$ then $a * b \in C_{\mathrm{c}}(G)$ and .
(2) The operations (1) and (2) make $C_{\mathrm{c}}(G)$ into a complex *-algebra.
(3) The algebra $C_{\mathrm{c}}(G)$ is unital if and only if G is discrete.
(4) If $a, b \in C_{\mathrm{c}}(G)$ then $\|a * b\|_1 \leq \|a\|_1 \|b\|_1$.

Part (4) is written out in my lecture notes. Prove it without looking there.

Second countability is not needed, but one must be more careful with the measure theory.

Problem 5. Let G be a group with the discrete topology. Let $C_{\mathrm{c}}(G)$ be as in Problem 4. Prove that $C_{\mathrm{c}}(G) \cong \mathbb{C}[G]$.

Problem 6. Let $(A_i)_{i \in I}$ be a family of C*-algebras. Using the usual set theoretic product, set
$$A = \left\{ a \in \prod_{i \in I} A_i \colon \sup_{i \in I} \|a_i\| < \infty \right\}.$$
For $a \in A$, define $\|a\| = \sup_{i \in I} \|a_i\|$. Prove that A is a C*-algebra and that B is the product of the C*-algebras A_i in the sense of category theory. (That is, prove that A satisfies the appropriate universal property.)

Because of the universal property, one sometimes writes $A = \prod_{i \in I} A_i$. This notation must be used with caution, because of conflict with usual set theoretic product.

Problem 7. Let A and B be unital C*-algebras, and let $T \colon A \to B$ be unital and positive. Prove that $\|T\| = 1$.

Problem 8. Let A and B be C*-algebras, and let $T \colon A \to B$ be positive.

(1) If B is commutative, prove that T is completely positive. (It suffices to take $B = \mathbb{C}$.)
(2) If A is commutative, prove that T is completely positive.

Problem 9. Prove that the transpose map $M_2 \to M_2$ is positive but not completely positive.

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