Math 680 W111

Thm (C*-Thm). (4.49 in book) If \( X \) is connected, the trace \( \tau \)

is a bijection from open subalgebras \( U \subset \mathcal{A} \) to closed ideals \( \mathcal{I} \subset \mathcal{C}(\mathcal{A}) \) given by \( U \mapsto \mathcal{I}(U) = \{ f \in \mathcal{C}(\mathcal{A}) : f = 0 \text{ off } U \} \) (closure of \( \mathcal{I}(U) \)

but \( \tau^{-1}(U) = \mathcal{I}(U) \).

Results: Closed sets \( \mathcal{C}(\mathcal{A}) \) yield \( \mathcal{I} = \{ f \in \mathcal{C}(\mathcal{A}) : f|_{\mathcal{I}} = 0 \} \).

Now suppose \( \mathcal{A} \) is a commutative Banach algebra with maximal ideal space \( \text{Max}(\mathcal{A}) \).

For \( I \subset \text{Max}(\mathcal{A}) \) closed, \( \mathcal{I} = \{ f \in \mathcal{A} : f|_{\mathcal{I}} = 0 \} \) a closed

ideal in \( \mathcal{A} \), different for different \( I \).\( \mathcal{I} \) is called kernel of \( F \)

\( \ker(F) = \{ \mathcal{I}(I) : \text{but } \mathcal{I} \neq \mathcal{I}(I) \} \).

For a graded \( \mathcal{A} \) "spectral for \( \mathcal{A} \)," \( \mathcal{I} \subset \text{Max}(\mathcal{A}) \).

Here with closed ideal in \( \mathcal{A} \).

Ex (not related to groups). Let \( \mathcal{A} = \mathbb{R}^G \) have \( D = \{ e \in \mathbb{C}^G : |e| = 1 \} \)

and \( A = D \). Hence \( D = \{ e \in \mathbb{C}^G : |e| = 1 \} \).

For \( \mathcal{I} \in \text{Max}(\mathcal{A}) \) and \( \mathcal{I} \neq \mathcal{I} \).

\[
\mathcal{I}(\mathcal{I}) = \{ f \in A : f|_{\mathcal{I}} = 0 \} = \mathcal{I}.
\]

Motivation: \( \text{Max}(\mathcal{A}) = D \).

Set \( \mathcal{I} = \{ f \in A : f|_{\mathcal{I}} = 0 \} \).

This is a closed ideal. All of these are distinct but all are equal with the closed set \( \mathcal{I} \).

In general, for \( U \subset \text{Max}(\mathcal{A}) \) closed ideal, the \( \mathcal{I} \subset \text{Max}(\mathcal{A}) \) \( \mathcal{I}(U) \) closed ideal in \( \text{Max}(\mathcal{A}) \).

\( \ker(F|_U) = \mathcal{I}(U) \).

For a particular \( \mathcal{I} \), the \( \mathcal{I} = \mathcal{I}(U) \).

If \( \mathcal{I} \) is not abelian, then \( L^1(A) \) has spectral synthesis.

In general, every \( \mathcal{I} \subset \text{Max}(\mathcal{A}) \) has spectral synthesis.\( L^1(A) \.

Ch. For even \( G = \mathbb{Z} \) and \( p \in (1, \infty) \), look at the left regular rep of \( (0, \infty) \) in \( L^p(\mathbb{Z}) \).

Call it \( F_p(\mathbb{Z}) \). For \( p = 2 \), get \( C(\mathbb{S}) \). Generally \( C(\mathbb{S}) \).

For \( p = 1 \), get \( L^1(\mathbb{Z}) \).

For \( p \in (1, \infty) \), \( L^p(\mathbb{Z}) \) does this happen have spectral synthesis? Noting these. Might be hard.
Consider the set $\mathbb{G} = \{g \in L(V, \mathbb{H}) : g(\alpha) \in V \text{ for all } \alpha \in V\}$. The set $\mathbb{G}$ is a vector space under the operations of addition and scalar multiplication.

Recall that $C(\mathbb{G}) = \{f : \mathbb{G} \to L(V, \mathbb{H}) \text{ such that } f(g)(\alpha) = g(\alpha) \text{ for all } g \in \mathbb{G}, \alpha \in V\}$. The set $C(\mathbb{G})$ is a vector space under addition and scalar multiplication.

For $f, g \in C(\mathbb{G})$, let $f + g$ be the vector defined by $(f + g)(g)(\alpha) = f(g)(\alpha) + g(\alpha)$ for all $g \in \mathbb{G}, \alpha \in V$.

Let $\alpha \in V$ and $\beta \in \mathbb{H}$. Then $f(\alpha)(\beta) = \sum_{i=1}^{\dim V} a_i g_i(\alpha)(\beta)$ for some $a_i \in \mathbb{H}$. The set $C(\mathbb{G})$ is closed under addition and scalar multiplication.