Lemma. \( G \) opt, \( f_1 + f_2 : G \to G \) below, \( f_1 = f_2 \text{ a.e.} \). \( f_1(gh) = f_2(hg) \) for a.e. \( (g,h) \in G \times G \).

Then \( f_1(gh) = f_2(hg) \) for a.e. \( (g,h) \in G \times G \).

Proof: assuming \( G \) is 2nd oth., so can use Borel sets. (not results).

If \( G \) assumes we have \( G \). Borel set \( M \). \( G \) a.e. \( \mathcal{N} \) and \( f_1 = f_2 \) a.g. \( G \times \mathcal{M} \).

Define \( R = \{(g,h) \in G \times G : gh \in \mathcal{M}\} \). \( S = \{(g,h) : h \in \mathcal{M}\} \), and

\( \mathcal{N}_1 = \{(g,h) : f_1(gh) \neq f_2(hg)\} \). By \( \text{hyp}, \ (\mu, \nu)(\mathcal{N}_1) = 0 \).

Also (which): \( \mathcal{N}_2 \subseteq \mathcal{N}_1 \cup \mathcal{N} \times \mathcal{N} \). So enough to show \( \mu(\mathcal{N}_1 \cap \mathcal{N}_2) = 0 \).

If \( \mathcal{N}_1 \cap \mathcal{N}_2 \) see the same: only do first. Use Fabini. M Borel so \( R \approx \text{Borel} \), so Fabini applies. For \( g \in G \), \( \mathcal{S} \not\subseteq G \) \( (g,h) \in G \). \( S \not\subseteq \mathcal{M} \) which has measure 0.

So Fabini says \( \exp(R) = 0 \).

Remark. \( G \) opt. Then \( \mathcal{L}^1(G) \), \( f \to \mu(G \mid \mathcal{F}), \) is a Banach alg under convolution, similarly for \( \mathcal{A} \).

\[
\text{[For } p \neq 1, \text{ don't get bdd approx identities.]}
\]

Proof: For \( \mathcal{L}^p(G) \), need \( \mu(G) = 1 \), so \( p_1 \leq p_2 \Rightarrow \|f\|_{p_1} \leq \|f\|_{p_2} \). Thus \( f_1, f_2 \in \mathcal{L}^p(G) \Rightarrow f_1 \ast f_2 \in \mathcal{L}^p(G) \), and \( \|f_1 \ast f_2\|_p \leq \|f_1\|_p \|f_2\|_p \).

For \( \mathcal{A} \) need to know \( f_1, f_2 \text{ cont} \Rightarrow f_1 \ast f_2 \text{ cont.} \) (See Gilard, Prop 2.43)

Recall notation: spaces of central tests [an ideal for us] are \( \mathcal{Z} \mathcal{L}^0(G), \mathcal{Z} \mathcal{C}(G) \).

Lemma. For \( G \) opt, \( p \in [1, \infty] \), \( \mathcal{Z} \mathcal{L}^p(G) \) is the center of the Banach alg \( \mathcal{L}^p(G) \).

\( \mathcal{Z} \mathcal{C}(G) \) is the center of \( \mathcal{C}(G) \) with convolution.

Proof. Do \( \mathcal{L}^p(G) \) first. Let \( Z \in \mathcal{L}^p(G) \). Then \( Z \) is in the center if \( \mathcal{A} \), \( \mathcal{A} \) of \( \mathcal{L}^p(G) \).

One has \( Z + f = f + Z \), that is, for a.e. \( g \in G \), \( \int_G Z(h)^{-1} v(g) \, dP(h) = \int_G f(h) Z(h^{-1}) \, dP(h) \).

Choose \( v \) on both to get \( Z(h) = \int_G Z(hy) \, dP(y) = \int_G Z(hy) \, dP(h) \) (why?

If \( Z(h) = Z(hy) \) for a.e. \( (g, h) \in G \times G \), then, by Fabini, for a.e. \( g \), one has \( Z(h) = Z(hy) \) for almost every \( h \), so for those \( g \), (4) holds for any \( f \).

Thus, \( Z \) is in the center.

For reverse, problem: the set of \( g \) for which (4) holds depends on \( f \).

Rewrite: \( \int_G (Z(gh) - Z(wh)) f(wh) \, dP(h) = 0 \) \( \forall f \in L^1(G) \).\]
In (4.4), replace \( f \) with \( h \mapsto f_h \) which sends \( L^1(G) \to L^p(G) \). (using \( L^1 \sim L^p \))

So assume that \( \forall f \in L^p(G) \) \( \exists N \subseteq G \) with \( \mu(N) = 0 \) s.t. \( \forall g \in N \Rightarrow \int_G (z(gh) - z(hg)) f(h) d\mu(h) = 0 \).

The assignment \( g \mapsto \left[ \text{for } h \mapsto z(gh) \right] \) is continuous \( G \to L^1(G′) \). Using \( L^p(G) \subseteq L^1(G) \) similarity with \( z(hg) \).

Thus \( g \mapsto \int_G (z(gh) - z(hg)) f(h) d\mu(h) \) is cont. whenever \( f \in L^1(G) \).

Since \( L^1(G) \subseteq L^p(G) \) \( \mu(g) = \text{zero for } \forall g, \text{ for } f \in L^1(G) \). \( \mu_g = \phi \).

Now by Riesz rep., \( \forall g \), the \( \text{for } h \mapsto z(gh) - z(hg) \) is zero a.e.

If take \( G = \mathbb{R} \) (which we can do using prev. lemma), have the measurability needed to use Fabius \( \alpha \) and conclude \( z(gh) = z(hg) \) for a.e. \((g, h) \in G \times G . \)

This is the \( L^p \) case.

For \( C(G) \). If \( z \) is a constant function, then \( z \) is in the center of \( \int \) \( (G) \)
by prev. case, and \( C(G) \subseteq L^1(G) \), so \( z \) is in the center of \( C(G) \).

Conversely, similarly to above. Enough to show: \( \int G (z(gh) - z(hg)) f(h) d\mu(h) = 0 \)
for all \( h \in C(G) \).

(We don't need to worry about \( N \) as above, since now every thing is cont.)

Sending \( g \) to the \( \text{for } h \mapsto z(gh) - z(hg) \) is cont. \( G \to L^1(G) \).

Also \( C(G) \) is dense \( \subseteq L^1(G) \). So we have \( \left< f_g, f \right> = 0 \) \( \forall h \in C(G) \).

Thus \( \forall h \text{ is the zero function in } L^1(G) \).

Use Fabius, \( z(gh) = z(hg) \) for a.e. \((g, h) \in G \times G . \)

But this is cont. in \((g, h) \), so \( z(g) = z(h) \) for all \( g, h \in G \).

At least if \( G = 2nd \) Alex, then \( \text{for } f \in \mathcal{D} \), \( \text{mable with } f_gh = f(hg) \) \( \forall \alpha \) \( \in \mathcal{D} \).

(We need this above and approximate identities in \( \mathcal{D}(G) \) for \( L^1(G) \).)