Recall: $\text{SU}(2) = \{ u(\alpha, \beta) : \alpha, \beta \in \mathbb{C}, \alpha \beta + 1 = 1, u(\alpha, \beta) = (\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}) \}$. 

$\pi : \text{SU}(2) \to \text{U}(S^2)$ is the identity repn.

\[ \text{Note: } \text{Follows } \pi, \text{ is the compatible rep of the representation.} \]

Let $\mathcal{P}$ be the set of polynomials in two variables: $\sum_{k=0}^{\infty} c_k x^k y^l$ (where $c_k, c_l \in \mathbb{R}$). Identify the with those in $S^2 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$ with $c_k = 0$ for all but finitely many $k$.

Let $C_m$ be the subset consisting of the ones which are homogeneous of degree $m$. Then $C_m \subseteq S^2$, $\dim C_m = m + 1$.

To see it exists: Let $m$ be a positive integer. Then, $\mathcal{P} \subseteq \mathcal{C} \subseteq \mathcal{C}$, $\mathcal{C} \subseteq \mathcal{L}^2(S^2)$, $\mathcal{L}^2(S^2)$ is a Hilbert space.

Then $\mathcal{P} \subseteq \mathcal{C} \subseteq \mathcal{L}^2(S^2)$.

$\mathcal{P} \subseteq \mathcal{C}$ is the closure in $\mathcal{L}^2(S^2)$.

Note: $\mathcal{P}$ is closed since it is finite dimensional.

Define a unitary rep $\pi : \text{SU}(2) \to \text{U}(L^2(S^2))$ using entries of $\text{SU}(2)$ on $S^2$.

Then $\pi(w) \pi(w)^* = 1$. We will use $\pi^{-1}(w) = \pi(w^*) = (w^*)^{-1}$.

Note: $\pi(w, w) = \pi(w \bar{w})$. Check:

\[ \pi(w, w)^* = \pi(w^\dagger, w^\dagger) \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right). \]

Thus the subspaces $C_m \subseteq \mathcal{P}$ are invariant. Take $\pi_m = \pi^m$ for $m = 0, 1, 2, \ldots$.

We will show that $\pi_m$ are inequivalent for different $m$, and that they exhaust $\text{SU}(2)$.

They are inequivalent since $\text{deg } C_m = m + 1$. 

$\pi_0$ is the 1-dim trivial repn.

Claim: $\pi_1$ is the identity repn (which we already called $\pi$, above).

Noting $\pi_0$ is trivial, eq $\pi_1$.

Hf of claim: Consider $p(5, w) = \overline{5}, q_5(5, w) = w$. These are a basis for $\mathcal{F}$.

Consider $C^2 \to p_1 : (1, 0) \mapsto p(0, 1) \mapsto q_5$. This is invertible. (Maybe not entirely, but $\mathcal{F}$ is not finite dimensional.)

Thus $\pi$ is unitary. For $(0, 1)$ and $q_5$, it also works (constant). Claim proved.
Lemma: Let $f: \mathbb{C}^2 \to \mathbb{C}$ be continues and $m$-homog. \((f(\lambda \mathbf{z}) = \lambda^m f(\mathbf{z}) \text{ for } \lambda \in \mathbb{C}, \mathbf{z} \in \mathbb{C}^2)\), with $m \geq 0$. Then \(\int_{\mathbb{S}^2} f(\mathbf{z}) \, d\mathbf{z} = \frac{1}{\pi^2} \int_0^{\infty} \int_{\mathbb{S}^2} f(\mathbf{z}) e^{-\|\mathbf{z}\|^2} \, d\mathbf{z} \, dm_{\mathbf{z}}(\mathbf{z})\).

Recall: \(\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt \quad \text{(for Re}(z) > 0) ; \quad \Gamma(z) = \Gamma(z+1) \quad \text{for} \quad \text{integer} \quad z \geq 1\).

Generally: \(\zeta(2) = \pi^2 / 6\).

Comment: One should expect this with some constant.

Proof: Use generalized polar words. Fubini:
\[
\int_{\mathbb{S}^2} f(\mathbf{z}) e^{-\|\mathbf{z}\|^2} \, dm_{\mathbf{z}}(\mathbf{z}) = c \int_0^{\infty} \left( \int_{\mathbb{S}^2} f(\mathbf{z}) e^{-r^2} r^3 \, d\mathbf{z} \right) \, dr
\]
for some \(c\) (to be calculated constant \(c\)).

This is a problem in Ch 2 of Rudin, Real and Complex Analysis.

\[
= c \left( \int_0^\infty r^{m-3} e^{-t-r^2} \, dr \right) \left( \int_{\mathbb{S}^2} f(\mathbf{z}) \, dm_{\mathbf{z}}(\mathbf{z}) \right)
\]
using \(m\)-homogeneity of \(f\).

Change variables: \(t = r^2\), get
\[
\int_0^\infty t^{m/2} e^{-t} \frac{1}{2} t^{-1/2} \, dt = \frac{1}{2} \Gamma \left( \frac{m}{2} + 2 \right).
\]

So,
\[
(\ast) \quad \int_{\mathbb{S}^2} f(\mathbf{z}) e^{-\|\mathbf{z}\|^2} \, dm_{\mathbf{z}}(\mathbf{z}) = \frac{c}{2} \Gamma \left( \frac{m}{2} + 2 \right) \int_{\mathbb{S}^2} f(\mathbf{z}) \, dm_{\mathbf{z}}(\mathbf{z}).
\]

In (\ast) put \(f = 1\), so \(m = 0\), and get
\[
c^2 \left( \int_0^\infty e^{-t} \, dt \right)^2 = \frac{c}{2} \Gamma(2) \frac{1}{2} \int_{\mathbb{S}^2} \, dm_{\mathbf{z}}(\mathbf{z}) = \frac{c}{2} \Gamma(2) \pi^2 (S^2) = \frac{c}{2} \pi^2.
\]
So \(c = 2\pi^2\) (this is the unnormalized surface measure of \(S^2\)).

Put this in (\ast) & rearrange to get result. END

There are analogs on \(\mathbb{H}^d\) for any \(d\). True condition on \(m \geq 2\) (with \(f(0)\) not defined if \(m < 0\)).
\[ \text{Lemma: Let } p, q, r, s \in \mathbb{Z}_0. \text{ Then} \]
\[ \int_{S^3} \xi^p \bar{\xi}^q \omega \bar{\omega}^s \, d\sigma (\xi, \omega) = \begin{cases} 0 & (p, r) \neq (q, s) \\ \frac{p! r!}{(p + r)!} & (p, r) = (q, s) \end{cases} \]

\[ \text{Pf:} \text{ Integral is harmonic of deg } p + q + r + s. \text{ So need only find} \]
\[ \int_{S^2} \xi^p \bar{\xi}^q \omega \bar{\omega}^s \omega^{-2} \, d\mu_4 (\xi, \omega) \]
\[ = (\int_{S^2} \xi^p \bar{\xi}^q \omega^{-2} \, d\mu_4 (\xi, \omega)) (\int_{S^2} \omega \bar{\omega}^s \, d\mu_2 (\omega)). \]

Compute first (also second) in polar coords:
\[ \int_0^\infty \int_0^{2\pi} e^{i(p-q)\rho \theta} \rho^{p+q} \, d\rho \, d\theta = \frac{2\pi}{2^{p+q} (p+q)!} \]
\[ \iff \rho = q \text{ done. If } p = 0, \text{ get } 2\int_0^\infty r^{2p+1} e^{-r^2} \, dr. \text{ Calculate by} \]
\[ \text{change of variable } r = t^{1/2} \text{ as in proof proof.} \]

To get desired in lemma for \((p, r) = (q, s)\), combine the estimates from here and previous lemma, and rearrange. \(\blacksquare\)

\[ \text{Check: For } (q, s) \in (\mathbb{Z}_0^2 \setminus \{(0,0)\}), \text{ can consider the functional on } L^2(S^3) \]
\[ \text{given by } f \mapsto \int_{S^2} f(\xi, \omega) \xi^q \bar{\xi}^s \, d\sigma (\xi, \omega). \text{ This resides on } L^2, \text{ but} \]
\[ \text{is not the zero functional. Also use } \xi^q \bar{\xi}^s \text{ on } S^3. \]

This shows \(L^2 \supset L^2(S^3). \)