Def: Let \((X, \mathcal{M})\) be a measurable space \((M \in \mathcal{M} \text{ m.e.a})\). A measurable field \(\Gamma\) of Hilbert spaces consists of a family \((H_x)_{x \in X}\) of separable Hilbert spaces and a collection \(\mathcal{F}(H)\) of what will be called measurable sections \((\text{fin} \subseteq m \times \text{fin} \subseteq \mathcal{H}, \mathcal{A} \subseteq \mathcal{H}, \forall x \in X)\) satisfying:

1. \(\mathcal{F}(H)\) is a weak subspace of \(\bigotimes_{x \in X} H_x\).
2. \(\forall \xi, \eta \in \mathcal{F}(H), \ x \mapsto \langle \xi(x), \eta(x) \rangle \) is a m.e. fun.
3. There is a seq \((\psi_n)\) in \(\mathcal{F}(H)\) s.t. \(\text{Span} (\{\psi_n(x) : n=1, 2, \ldots\})\) is dense in \(H_x\).
4. Let \(x \in \bigotimes_{x \in X} H_x\). Suppose \(A \in \mathcal{F}(H)\), \(x \mapsto \langle \xi(x), \eta(x) \rangle\) is m.e. \(\Rightarrow \exists \xi \in \mathcal{F}(H)\).

Prop. (7.19 in Folkard). Let \((H_x)_{x \in X}\) be a family of sep Hilbert spaces over \((X, \mathcal{M})\).

Let \((\xi_n(x))_{n \in \mathbb{N}}\) be a seq of sections st \(x \mapsto \langle \xi_n(x), \eta(x) \rangle\) m.e. \(\mathcal{A} \in \mathcal{M}\) \(\Rightarrow \exists \eta(x) \in H_x \forall x \in X\).

Then:

1. \(x \mapsto \langle \xi(x), \eta(x) \rangle\) is m.e.
2. \(\exists \xi \in \mathcal{F}(H)\) s.t.
3. \(\forall \eta \in \mathcal{F}(H) \Rightarrow \forall \xi \in \mathcal{F}(H) \Rightarrow \xi \perp \eta\).

Theorem: For any seq \((\eta_n)\) of sections st:

1. \(\forall x \in X, \ (\eta_n(x))_{n \in \mathbb{N}}\) is an ONB in \(H_x\).
2. \(\forall n \in \mathbb{N}, \xi \in X \Rightarrow \xi \perp \eta_n(x)\).

Then: \(\exists \xi \in \mathcal{F}(H)\) s.t. \(\forall \eta \in \mathcal{F}(H) \Rightarrow \xi \perp \eta\).

Temporary Terminology: Call \([(H_x)_{x \in X}, \ (\xi_n)_{n \in \mathbb{N}}]\) as a hypernetes a skeleton

for a m.e. field. Require \(\forall x, \exists \xi_0, \eta_0 \in H_x\) has dense span and \(\forall \mathcal{M} \in X \times \mathcal{M}\).

Further, if \(S \subseteq \bigotimes_{x \in X} H_x\), then say a sectio \(S \in \mathcal{F}(H)\) is in the measurable span of \(S\) if there is a m.e. partition \((X_k)_{k \in \mathcal{K}}\) of \(X\) and for each \(k\), \(r(k) \geq 0\),

\(\eta_1, \ldots, \eta_n(x) \in S\), m.e. sectio \(f_{(k)} \in X_k\), \(f_{(m)} \in X_k\) m.e. sectio \(g_{(k)} \in X_k\), \(\forall x \in X_k\), \(\forall \mathcal{M} \in \mathcal{M}\) \(S(x) = \sum_{k \in X} f_{(k)}(x) \eta_{(k)}(x)\).

The conclusion \((2.2)\) of prop says \(\eta_1, \ldots, \eta_n\) are in m.e. span of \(\{\eta_1, \ldots, \eta_n\}\).

Will do with \(S_0, \ldots, \eta_n\) are in m.e. span of \(\{\eta_1, \ldots, \eta_n\}\).

For \(S\) we can (but not have):

1. \(\forall S_0, \ldots, \eta_n \in S\), \(x \mapsto \langle S(x), \eta(x) \rangle\) is m.e.
Lemma. Suppose $S \subseteq E$ satisfies: $x \mapsto \langle \delta_x, \eta \rangle$ is m.b. $\forall \eta \in S$. Then also $x \mapsto \langle \delta_x, \eta \rangle$ is m.b. for all $\eta$ in the m.b. space of $S$.

Proof: clear.

Cor. If $S$ satisfies $(\star)$, so does its m.b. span. Pf. apply lemma twice (once in $\mathbb{C}^k$).

Lemma. Suppose $S$ satisfies $(\star)$. Let $\xi_1, \ldots, \xi_n \in S$. Then $\sum_{i=1}^n \xi_i = 0$ if and only if $\xi_i$ are li indeps.

Proof. Use: $\xi_1, \ldots, \xi_n$ are li indeps iff dot $\left( \frac{\langle \xi_1, \xi_1 \rangle}{\langle \xi_1, \xi_1 \rangle}, \ldots, \frac{\langle \xi_n, \xi_n \rangle}{\langle \xi_1, \xi_1 \rangle} \right) \neq 0$. \hfill \Box

Will also need: If $\xi_1, \ldots, \xi_n$ are li indeps, $\eta$ arbitrary in span($\xi_1, \ldots, \xi_n$), then $\eta$ can be written of the form $\eta = \sum \lambda_i \xi_i$.


classic of Prop. Suppose $(\mathcal{O}_X, (\mathcal{F}_i)_{i \in \mathbb{Z}_2})$ is a skeleton for a m.b. field $\mathbb{K}$.

1) $\forall x, \xi \in X$, dim $(\mathbb{K}_x) = d_x$ is m.b.

2) There are $\eta_1, \eta_2, \ldots$ in the m.b. span of $\mathbb{K}_{\eta_1}, \mathbb{K}_{\eta_2}, \ldots$ s.t. $\forall x \in X$, if dim $(\mathbb{K}_x) = d_x$ then $\eta_1(x), \eta_2(x), \ldots$ form an ONB of $\mathbb{K}_x$ and $(\mathbb{K}_x \cap \mathbb{K}_{\eta_i}) = \{0\}$ for $i = 1, 2, 3, \ldots$.

Proof: Construct partitions etc by induction, to begin with using "basis" instead of ONB in conclusion.

Define $\eta_i(x) = \sum \xi_k \eta_i(x)$ with $\xi_k$ the least $k$ s.t. $\xi(x) \neq 0$. [If no such $k$, take $\eta_i(x) = 0$.]

Given $\eta_1, \eta_2, \eta_3$, define $\eta_{123}(x) = \sum \eta_{k}(x)$ with $\xi_k$ the least $k$ s.t. $\xi_k(x)$ is not in span $(\eta_1(x), \ldots, \eta_3(x))$, $\eta_{123}(x) = 0$ if no such $k$ exists.

Claim: $\forall x$ in m.b. partition $(x^{(i)}_{k})_{k \in \mathbb{Z}_2}$ s.t. where in $S_2$, then $\mathbb{K}_x$ either $\mathbb{K}_x(0) = \mathbb{K}_x(1)$ or

$\mathbb{K}_x(0) \in \mathbb{Z}_2$. If $x^0(x) = \sum \xi_k(0) \in \mathbb{K}_x(0)$ for all $x \in X^{(i)}_{k}$. Moreover, $(X^{(i)}_{k})$ reduces $(X_{k})_{i = 0, 1}$.

Write to avoid special notation when only finitely many sets allow $x^{(i)}_{k} = \mathbb{K}_x(0)$.

Proof of claim: Induction on $n$.

Set $\tilde{N}_1 = \sum x \in X : \langle \tilde{x}, \eta \rangle \neq 0$, m.b.,

Set $X^{(i)}_k = N_k \setminus \cup_{j = 1}^{\kappa - 1} N_j$ for $k = 1, 2, \ldots$ and $X^{(i)}_0 = X \cup \cup_{j = 1}^{\kappa - 1} N_j$. These are m.b., and $\tilde{N}_i \subseteq X^{(i)}_k$ for $k \neq 0$, $\eta \neq 0$ on $x^{(i)}_k$. This does not.
Suppose $X$ is finite.

**Induction step:** Consider only $Y = X^{(n)}$ for some $k \leq \mathbb{Z}_2$. Get partitions of each of these.

The partition of $X$ will be the collection of all sets in and at these partitions.

**Case 1:** $\eta_1 = 0$ in $Y$. Then $\eta_{n+1} = 0$ in $Y$. Only need one set in partition.

**Case 2:** There are $k_1 < k_2 < \ldots < k_n$ s.t. $\eta_m = \delta_{m,n}$ in $Y$ for $m = 1, \ldots, n$.

Put set $N_k = \{x \in X^0 : (n_1, w_1, \ldots, n_k, w_k) \text{ are bi-indep} \}$ for $k \geq k(n)+1$.

Define $Y_k = \bigcup_{m=k(n)+1}^{k(n)} N_m$ for $k = k(n)+1, k(n)+2, \ldots$, and $Y_k = X \setminus \bigcup_{k=k(n)+1}^{k(n)+1} N_k$.

These are all noble. Check condition easily. Claim proved.

To finish proof: (1) follows from dim$(H_0) = 1$ for $\eta_1 = 0$, a noble condition.

(2) Need only get bi-independence. Apply Gram-Schmidt, which is correct.

Linearly indep $(\eta_1, \ldots, \eta_n)$ in $X$.

**Glimpse:** $\eta_1, \ldots, \eta_n$ is bi-indep.

**Glimpse:** $(\eta_1, \ldots, \eta_n)$ is bi-indep.

**Glimpse:** $\eta_1, \ldots, \eta_n$ is in a noble space.

**Glimpse:** The coefficients are not unique, but the eichroths give a choice of $(\lambda_1, \ldots, \lambda_n)$ which depends continuously on the family $(\langle \eta_1^*, \eta_i^* \rangle)_{i=1}^n$.

To begin with: $\lambda_1 = \langle \eta_1^*, \eta_1 \rangle$.

Then help $X \rightarrow \mathbb{R}$ etc are inverse image of in open set is noble.

That is $\eta$ only in $\mathbb{R}$ etc is the blank sets, not the false noble sets.

Suppose $f : X \rightarrow \mathbb{R}^n$ noble, $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ wrt. Then with $h f$ noble.

$h$ open implies $h^{-1}(U)$ open implies $\bigcup_{i=1}^n h^{-1}(U_i)$ open implies $(h f)^{-1}(U)$. 