Math 686

Existence of $L^2$ decomposition:

Given $\mathcal{H}$, then every rep in a $\mathcal{H}$ of unit norm, so a direct integral. Let $\mathcal{H}$ be a special case.

Gabrielian: Reps of $G$ are the same as irreducible reps of $C^*(G) \cong C_{0}(\mathcal{G})$.

Follow how in each term on reps of $C^*(G)$, which applies to $C_{0}(\mathcal{G})$ by unitarity.

It says (in 2nd sense case) a rep $\pi$ of $C^*(G)$ is a direct sum (in weak) of reps of the following form:

Two Borel measure $\mu$ on $X$ and look at multi-op on $L^2(X,\mu)$. Then reps are $\int_{X} \phi(x) d\mu(x)$.

Thus (following Arvind's book), let $(X,\mu)$ be a finite measure space. Let $\mathcal{H}_{0}$ be a sep Hilbert space.

Set $\mathcal{H} = L^2(X,\mu, \mathcal{H}_{0})$ (Scars of trivial noble field). Let $M \subseteq L(H)$ be the alg of mult; yes by $f \in L^\infty(X,\mu)$ [call $\Lambda$ the map $m(f)$]. Let $\mathcal{H}$ be the set of yes $\int_{X} \phi(x) d\mu(x)$ for herdo noble fields of operators $\phi \in L(H)$, [called decomposed operators]. Then $\mathcal{D} = \mathcal{M}$.

Claim: We did this when $\mathcal{H}_{0} = l_{2}$, so that it says $\mathcal{M} = \mathcal{M}$.

If $\mathcal{D} = \mathcal{M}$ trivial. Reverse: let $\mathcal{D} \subseteq \mathcal{M}$ and assume $a \in \mathcal{M}$. Assume if not take $a \notin \mathcal{M}$, else use $\mathcal{D}$ for the constant func in $\mathcal{H}$ with value $\frac{1}{2}$ (set is in $\mathcal{H}$ since $\mu$ is finite).

For each $a \in \mathcal{M}$, consider $a_{\mathcal{M}} \subseteq \mathcal{M}$. Choose a suitable noble field $k_{\mathcal{M}} : X \rightarrow \mathcal{M}$ noble where others $\phi$.$\Lambda$.

Claim: $\forall \xi, \eta \subseteq \mathcal{H}$, $| \langle k_{\mathcal{M}}(x) \eta \rangle | \leq ||a||_{2} ||\xi||_{2} ||\eta||_{2}$.

To prove this, first show: $f, g \in L^\infty(X,\mu) \Rightarrow$

(a) $\int_{X} \langle k_{\mathcal{M}}(x) \eta \rangle f(x) g(x) d\mu(x) \leq ||a||_{2} ||\xi||_{2} ||\eta||_{2} ||f||_{2} ||g||_{2}$.

An attempt to estimate norm of mult by $x \rightarrow \langle k_{\mathcal{M}}(x) \eta \rangle$ in $L^2(X,\mu)$.

To prove (a), let $\mathcal{M}$ be constant in $\mathcal{H}$ with value $\frac{1}{2}$ in $\mathcal{H}_{0}$.

Thus $\int_{X} \langle k_{\mathcal{M}}(x) \eta \rangle f(x) g(x) d\mu(x) = \int_{X} \langle f(x) k_{\mathcal{M}}(x) \eta \rangle d\mu(x)$

Since $a_{\mathcal{M}} \subseteq \mathcal{M}$.

Also now a cont. fcn in $X$.

So $||f||_{2} ||g||_{2} ||\xi||_{2} ||\eta||_{2}$.

So (a) is proved.

To get the claim: suppose $| \langle k_{\mathcal{M}}(x) \eta \rangle | > ||a||_{2} ||\xi||_{2} ||\eta||_{2}$ on some set $E$ then positive measure $\mu$ with $\epsilon > 0$. Consider $k_{\mathcal{M}} \eta$ supported in $B_{\mathcal{M}}(x, \epsilon)$ (done before), to get contradiction.
Choose a countable dense $(1 + i \alpha)\text{-vector subspace } V \text{ of } H_\alpha$. (Take a countable dense subset of \( H_\alpha \) and use \((1 + i \alpha)\span\{x\}\) for \( x \).)

Choose \( N_0 < X \) s.t. \( \mu(N_0) = 0 \) and \( \forall x \in X \setminus N_0, \forall \xi, \eta \in V, \langle k_\xi(\omega), \eta \rangle \leq \|\xi\| \|\eta\| \|W_\omega\| \).

Assume there are many \( \mu \)-sets of measure zero. For claim.

Since \( a \) is linear, \( \forall \lambda_1, \lambda_2 \in C \), \( \forall \xi_1, \xi_2 \in H_\alpha, \ k_{\lambda_1 + \lambda_2} = k_{\lambda_1} + k_{\lambda_2}, \text{ ae. } L^\infty(X) \).

So \( \exists N_1 < X \) with \( \mu(N_1) = 0 \), s.t. \( \forall \lambda_1, \lambda_2 \in \omega + \omega_1, \xi_1, \xi_2 \in V, \forall x \in X \setminus N_1, k_{\lambda_1 + \lambda_2}(x) = k_{\lambda_1}(x) + k_{\lambda_2}(x) \).

Set \( N = N_0 \cup N_1, \quad \mu(N) = 0 \).

For \( x \in X \setminus N \), we have a \((1 + i \alpha)\text{-sesquilinear form}\) on \( V \), given by

\[ \langle \xi, \eta \rangle \mapsto \langle k_\xi(\omega), \eta \rangle, \text{ for all } \langle \xi, \eta \rangle \in V \text{ with } \| \langle \xi, \eta \rangle \| \leq \|\xi\| \|\eta\| \|W_\omega\| \forall \xi, \eta \in V. \]

Extend by continuity to get an ordinary sesquilinear form \( \langle -,- \rangle_\omega \) on \( H_\alpha \), with

\[ \| \langle \xi, \eta \rangle_\omega \| \leq \|\xi\| \|\eta\| \|W_\omega\| \forall \xi \in X \setminus N \forall \eta \in H_\alpha. \]

A dty 3.2.1.52 space that gives \( a_x \in L(H_\alpha) \) s.t. \( \langle \xi, \eta \rangle_\omega = \langle k_\xi(\omega), \eta \rangle \forall \xi, \eta \in H_\alpha \).

One gets \( \langle a_x, \eta \rangle_\omega = \langle k_\xi(\omega), \eta \rangle \forall \xi, \eta \in V, \forall x \in X \setminus N. \)

Check now that \( x \mapsto a_x \) is a mbe family of operators. Set \( b = \sum \langle a_x, \cdot \rangle \).

Then \( \langle a_x, \eta \rangle = \langle k_\xi(\omega), \eta \rangle = \langle b_x, \eta \rangle, \forall \xi, \eta \in V, \forall x \in X \setminus N. \)

Note that this works with \( \xi, \eta \in H_\alpha \).

Finish pf by showing \( a = b \). Enough to consider the maps \( x \mapsto k_\xi(\omega) \) for \( \xi \in H_\alpha \) and \( \omega \in L^\infty(X) \).

Since these are dense in \( H_\alpha \).

Thus \( k_\xi(\omega) = m(f)k_\xi = m(f)b_x = / \text{ m(f)x} \)

\[ q \in M^1, \quad a \text{ agrees with } b \text{ in } \mu \text{-sets for } \mu \text{-meas } M \text{-sets with dty. integrals, } \mu \text{-meas } M \text{-sets.} \]

Claim: A n.dby \( M \) is typed i.f A n.dby \( M \) is typed i.f \( b \in Z(M) \), \( b \text{ is a n.dby abelian } \mu \text{-meas } M \text{-set. \quad q} \]

Here (i) if \( M \) is typed i.f \( M \) is typed i.f \( M \) is typed i.f 1.

By [Lem] is typed i.f \( p = 1 \), take \( \epsilon \) to be any real \( \lambda \) point.

\[ \lambda \in L(H) \text{ is typed i.f } p = 1, \text{ take } \epsilon \text{ to be any real } \lambda \text{ point.} \]

\[ \langle \xi, \eta \rangle \in L(H)^2 \text{ is typed i.f } H \text{ is already abelian.} \]

\[ M = L(H) \text{ is an abelian group } \forall \xi, \eta \in H \text{ that } \forall \xi, \eta \in H \text{ has } L(H) \text{-type } p \text{. If } \lambda \in \mathbb{C} \text{ then } \lambda \text{ is typed } = \text{ i.f } M \text{ is typed } p. \]

Then \( M \) is typed i.f \( M \) is typed i.f \( M \) is typed i.f 1.
A sep. (prob. not needed) Thus, the following are equivalent:

1. \( A \) is a type I C*-alg.
2. Every repn \( \pi \) of \( A \) is \( \pi(A) \) a type I \( \mathcal{N} \) alg.
3. Every irrept repn \( \pi \) of \( A \) on \( H \) has \( \pi(A) \subseteq K(H) \).
4. \( A^{**} \), the second dual of \( A \), is a type II \( \mathcal{N} \) algebra.

(There are more eq. conditions, less relevant to the question.)

Warning: When regarded as a C*-alg., \( L(\mathcal{N}^2) \) is not type II.

Then, if sep Hilbert space, \( M \subseteq L(H) \) type II \( \mathcal{N} \) alg. Then \( \exists \) direct sum decompo.

\[
H = \bigoplus_{n \in \mathbb{N}} H_n, \quad H_n \text{ M-inv subspaces, with projns } p_n \text{ onto } H_n.
\]

Such that \( p_n M p_n \subseteq L^\infty(X_n, \mu_n) \otimes M_0 \) for some finite measure space \((X, \mu, \mathcal{N})\).

Pf: \( n = \infty, M_0 = L(\mathcal{N}) \).

One gets ultimately (one does not need it) \( M = \mathcal{V} \mathcal{N} \bigotimes \text{ direct sum of } p_n M p_n \).

[It all holds says]