Proof.

Consider the locally compact group \( G \rightarrow \text{L(H)} \) with cyclic unitary representation \( M \subset C(G) = \pi(G)' \). There exists a standard measure space \( (X, \mathcal{S}, \mu) \) with \( \mu \) finite and a real field \( \mathbb{R} \) of Hilbert spaces over \( X \), with a b foul fied field \( \pi \) of unitary rep of \( G \), along with a unitary \( \eta : H \rightarrow \int_X H_x d\mu(x) \), such that:

1. \( \eta \pi_x(x) = \sum \eta(a) \pi(a) d\mu(a) \quad \forall a \in G \).
2. \( \eta \pi_x(f) \eta^* = \sum \pi_x(f(a)) \eta(a) d\mu(a) \quad \forall f \in L^1(G) \).
3. \( \eta M \eta^* \) is the alg of diagonal operators on \( \int_X H_x d\mu(x) \).

What we did \( \nu \) will be:

Choose a cyclic vector \( \xi \in H \) s.t. \( ||\xi|| = 1 \). Set \( \nu = \meas(M) \), i.e., \( M \) is identified with \( C(G) \).

For \( \eta \in L^p(G) \), define \( \eta \) as the measure \( \lambda \eta = \eta \xi \xi \) and set \( \phi \in A \cap \mathcal{S}, \phi \ni \xi, \xi \) where \( A \in \mathcal{S} \).

Claim 1. \( C(G) \Rightarrow \lambda \phi \xi \xi = 0 \quad \forall \phi \in \mathcal{S} \).

Claim 2. Let \( b \in M \). Then \( \lambda bs_0 \xi, \xi \ll \nu \) and \( b \rightarrow \text{R-N den.} \).

Claim 3. \( \lambda s_0 \xi, \xi \ll \nu \) for \( s_0 \in V \).

Pf of Claim 3: Enough to do for \( s_0 = b, s_0 = c_0 \).

Use \( \lambda bs_0, c_0 = \lambda bs_0, c_0 \ll \nu \) and Claim 2.

Claim 4. For \( b \in M' \), \( \exists ! T(b) \in \text{L(H)} \) s.t. \( \langle T(b) \xi, \xi \rangle = \int_X \delta_b d\lambda \xi, \xi \quad \forall \xi \in V \).

Moreover, \( ||T(b)|| \leq ||b|| \) and \( T \) is Hermitian.

Pf of Claim 4. Define \( I \) on \( V \) by \( I \xi, \xi = \int_X \delta_b d\lambda \xi, \xi \) (by Claim 3).

Claim 2 (corrected). Enough to show \( \eta \in C(G) \Rightarrow \int_X \eta f(x) ds_0, s_0 || \leq ||b|| \int_X \eta || f || d\mu \).

Take \( a \in M \), \( f = x \).

Polar decomposition: \( a = s(x^2) v_2 = sc_0 \quad \text{with} \quad c_0 \geq 0 \).

Also \( ||s_0|| = 1 \) and \( s_0 = c_0 \in M \). Now \( \delta_s = \delta_0 \).

\( ||s_0 s_0|| = ||s_0|| \delta_0 \).

\( ||\lambda s_0, s_0|| = \int_X <\xi, \xi, s_0 s_0 > = \int_X <s_0, s_0, s_0> = \int_X <s_0, s_0, s_0> \leq ||b|| ||s_0|| \delta_0 \).

Thus \( \lambda s_0, s_0 \ll \nu \), which is what we wanted. Claim 2 done.
Claim 6. If \( b \in E' \) then \( T(b) = b \). and \( \delta = \delta_\mu \), a.e. \( [\mu_b] \).

Proof. For \( a \in E' \),
\[
\int \hat{a} \delta_b \delta_{\mu_b} \, d\nu = \int \delta_{\mu_b} \hat{a} \delta_b \delta_{\mu_b} \, d\nu = \int \delta_{\mu_b} \delta_b \, d\nu = \int \delta_b \, d\nu_b.
\]

Since \( \hat{a} \) is an \( L^1 \) function, \( \int \delta_{\mu_b} \delta_b \, d\nu = 0 \). Thus \( \int \delta_b \, d\nu_b = 0 \), and \( \hat{a} \delta_b \delta_{\mu_b} \, d\nu = 0 \).

Claim 7. \( b \in E' \) if and only if \( \delta_b \delta_{\mu_b} \# \nu_b \). Pf. \( a \in E' \),
\[
\int \hat{a} \delta_b \delta_{\mu_b} \, d\nu = \int \delta_{\mu_b} \hat{a} \delta_b \delta_{\mu_b} \, d\nu = \int \delta_{\mu_b} \delta_b \, d\nu_b = \int \delta_b \, d\nu_b.
\]

Since \( \hat{a} \) is an \( L^1 \) function, \( \int \delta_{\mu_b} \delta_b \, d\nu = 0 \). Thus \( \int \delta_b \, d\nu_b = 0 \), and \( \hat{a} \delta_b \delta_{\mu_b} \, d\nu = 0 \).

Claim 8. \( \nu_b \) has full support. Pf. Need to show \( a \in E' \), \( \nu = \nu_b \) on \( \text{supp}(\nu) \Rightarrow a = 0 \).

For such a \( a \), set \( \nu = \gamma \nu \hat{a} \nu \) and \( \delta_g = \delta_{\mu_b} \). Then \( \nu \mu = \gamma \nu \hat{a} \nu \) and \( \frac{1}{2} \gamma \nu \hat{a} \nu = 0 \), by Claim 3. Then \( \int \hat{a} \delta_b \delta_{\mu_b} \, d\nu = 0 \), and \( \nu \) a.e. \( \Rightarrow a = 0 \). Claim proved.

It follows that \( T(b) \) is the unique \( \hat{a} \) in \( E' \) which agrees with \( \hat{a} \), a.e. \( \Rightarrow \delta_b = T(b) \).

Next, for \( b \in E' \), regarded as \( b : \text{V} \rightarrow 2^{\text{V}} \), an homomorphism, define \( \gamma_b : \text{L}^1(\mu) \rightarrow \text{L}^1(\mu) \) by \( \gamma_b(f)(x) = \int \text{V} \delta_b(x) \delta_{\mu_b} \, d\mu \).

Claim 9. \( \gamma_b \) is an \( L^1 \) homomorphism. Pf. \( a \in E' \),
\[
\int \hat{a} \gamma_b(f)(x) \, d\mu = \int \hat{a} \int \text{V} \delta_b(x) \delta_{\mu_b} \, d\mu = \int \delta_b \hat{a} \, d\mu = \int \hat{a} \, d\nu_b.
\]

Since \( \hat{a} \) is an \( L^1 \) function, \( \int \delta_b \hat{a} \, d\mu = 0 \). Thus \( \int \hat{a} \, d\nu_b = 0 \), and \( \hat{a} \gamma_b(f)(x) \, d\mu = 0 \).

Claim 10. For \( f \in \text{L}^1(\mu) \), \( \gamma_b(f)(x) = \int \text{V} \delta_b(x) \delta_{\mu_b} \, d\mu \) if and only if \( f \in \text{L}^1(\mu) \). So \( \gamma_b \) is a homomorphism.