Let $\lambda \to e_{\lambda}$ be a cat path of idempotents in $A$ (until $B$ make alg). Then 3 cat path
\[ \lambda \to z_{\lambda} \text{ in } A \text{ st. } z_{0} = 1 \text{ and } \lambda_{0} e_{\lambda} z_{\lambda} = e_{\lambda} \text{ for all } \lambda \in [0,1]. \]
Similarly for path of projns in a $C^{*}$-alg, get a unitary path
Nonunital case: get $e_{0} \in A^{+}$, and get $V : \lambda \to s_{\lambda}$ in $A$ st. $s_{0}^{*} s_{\lambda} = e_{0}$, $s_{\lambda} s_{\lambda} = e_{\lambda}$.
\[ C^{*} \text{-case} \]
Looked at, for $x_{0}$ small, both of $([0,1], A)$ given by $\lambda \mapsto e_{0}$, $\lambda_{0} \mapsto e_{\lambda}$.
These differ by less than $\epsilon$, so are similar. Can find subtile $0 = \lambda_{0} < \lambda_{1} < \cdots < \lambda_{k-1}$ and glue together.
Details: exercise.

Con homotopy idempotents are similar (as unital case), alg $M_{n}N$ eq in general.
(similarly for projns in $C^{*}$-alg a unitary eq).

For $f : \eta_{0} \to \eta_{1} : A \to B$ are homotopic homs, then $V(\eta_{0}) = V(\eta_{1}) : V(A) \to V(B)$.

[Homotopy invariance]

(Correspond topologists' proof: that $h_{\lambda} : V \to Y$ homotopic, $E$ vector bundle over $Y \Rightarrow$
$\tilde{h}_{\lambda}(E) \cong h^{*}(E)$.)

Semi-groups are hard to work with. [But for Cuntz semi-groups seem to be study with them.] [maybe]

We want to make a grad at of $V(A)$ loses information, but keep some of it by considering
ordered gps.

Then there is a functor $G$ (Grothendieck group construction) from abelin semi-groups to abelin gps,
say $S \to G(S)$ such where $S$ ab semi-group, $M$ abelian gp, $f : S \to M$ is a semi-group
hom, then $\exists ! g : G(S) \to M$ s.t. diagram commutes:

\[ \begin{array}{ccc}
S & \overset{f}{\rightarrow} & M \\
\downarrow & & \downarrow \text{g} \\
G(S) & \overset{g}{\rightarrow} & M
\end{array} \]

Will sketch pf; details: exercise.

You have maybe seen two related things already:

1. Grouped $\mathbb{Z}$ from $\mathbb{Z}_{\geq 0}$. (Special $\mathbb{Z}_{\geq 0}$ has cancellation.
2. Localizatn of a commutative unital ring at a multiplicative subset
   (most common case: complement of a prime ideal.) (more implicit because of
   special case: field of fractions of an integral domain.
   Construction: $G(S) = (S \times S) / \sim$, with $(r,s)$ intended to represent $r-s \in G(S)$
   (really $[r] - [s]$.)

\[ \text{ } \]
Define \((r_1, s_1) \sim (r_2, s_2)\) if \(r_1 + s_2 + t = r_2 + s_1 + t\) (without \(t\) it is the rearrangement of \(r_1 - s_1 = r_2 - s_2\) to eliminate - signs). Write \([r, s]\) for eq. class of \((r, s)\).

Define \([r_1, s_1] + [r_2, s_2] = [r_1 + r_2, s_1 + s_2]\)

\(\sim [r_1, s_1]\) The identity is given by \([s, s]\) for any \(s \in S\), (Semraps are not allowed to be empty) \(\sim [s, s]\) for any \(s \in S\). (Need not assume \(S\) is unital)

Fix \(t, s \in S\) and define \(b_{s,t}(s) = [s + b_0, b_1]\).

The most work is needed to show that \(\sim\) is transitive. Need to show, if \(r_1 + s_2 + t = r_2 + s_3 + t\) and \(r_1 + s_3 + u = r_2 + s_1 + u\), then \([r_1, s_1] \sim (r_2, s_2)\). Need \(v\) such that \(r_1 + s_3 + v = r_3 + s_1 + v\).

For the universal property: \(g : G(s) \rightarrow M\)

is given by \([\text{real} : S \rightarrow M\) semimap there, \(M abdeg\) \(g([r, s]) = f(r) - f(s)\). “\(\exists q\).”

\(\text{Def. (not standard notation, Murphy uses “} K_0(A)\text{”) A Banach algebra.} \(H(A) = G(V(A))\).

When \(A\) is unital, this will be \(K_0(A)\). Further define \(H(A)_+\) to be the range of \(V(A)\) in \(H(A)\)

\((\text{will be} K_0(A)_+ \text{ when } A \text{ is unital})\)

1. \(\exists \phi \text{ (must be real) \(V(\phi) \cong \mathbb{Z}_{\geq 0}\text{ via } p \mapsto \text{rank}(p)\).}

Thus \(H(\phi) = K_0(r))\) is \(\mathbb{Z}\) with rank part \(\mathbb{Z}_{\geq 0}\).

If \(e = e_{j}\) for pos \(p\) then \(\text{led} - \text{led} \rightarrow \text{rank} e - \text{rank} (u).

2. \(A = L(\ell^2). \quad V(A) \cong \mathbb{Z}_{\geq 0} \uplus \mathbb{Q}\) \(\text{via } [p] \mapsto \text{rank}(p)\).

\(\text{Reim: } M_n(A) \cong L(\ell^2(\ell) - \ell^2(\ell)) \text{ (\(\cong L(\ell^2)\))}, \text{ and pos} p, q \in L(\ell^2)\) are

\(M \text{-VN eq } \text{iff } \text{rank}(p) = \text{rank}(q)\).

For \(H(A)\): \(m_1 + q = m_2 + q\) for all \(m_1, m_2, q \in \mathbb{Z}_{\geq 0} \uplus \mathbb{Q}\).

So \(\text{in } B(L(\ell^2))\)

\(L_{m_1, q} = L_{m_2, q}\) for all \(m_1, m_2, q, n_2, n_2\), so \(H(A) = \mathbb{Q}\).

Also \(H(A)_+ = \mathbb{R}\).

Write: \(H(A) = 0, \text{ } H(A)_+ = 0. \text{ (by std above of notation)}\)

3. \(S\) is any vector set. Then \(V(L(\ell^2(S)))\) is the set of codim\(s < \text{card}(S)\).

But still \(H, L(\ell^2(S)) = 0.\) in any Hilbert space \(H\).

Part (ii) Proof: \(p \in M \text{-VN eq } q \text{ iff } \text{Ran}(p) = \text{Ran}(q). \text{ For } \Rightarrow:\)
Assume \( s^* s = \varphi \), \( s^* s + \varphi \). Then take the restriction of \( s \) to \( \text{Ran}(p) \) and its restriction to \( \text{Ran}(q) \) to get an isomorphism \( \text{Ran}(q) \to \text{Ran}(q) \).

Need to be sure \( s \in \text{Ran}(q) \). Well, \( s = s^* s = q s \), so \( s \in \text{Ran}(q) \).

The inverse is given by taking \( s^* \), restrict to \( \text{Ran}(q) \) and then restrict to \( \text{Ran}(p) \).

\[ \begin{align*}
&\text{Claim: } \exists \text{ iso } \psi : \text{Ran}(q) \to \text{Ran}(p), \text{ then} \\
&\text{If } H \to \text{Ran}(q) \twoheadrightarrow \text{Ran}(p) \quad \text{and} \\
&\psi \text{ restriction to } \text{Ran}(q) = s^*.
\end{align*} \]

This is essential. There are cases of non-dense

Banach spaces \( E \) with \( K_0(L(E)) \neq 0 \).

Hermite expression: If \( A \in L(E) \) and \( p, q \in A \) properly then \( p \circ q \) iff \( \text{Ran}(p) \subseteq \text{Ran}(q) \)

via an iso which is in \( A \). (exists \( A \) s.t. \( s \in \text{Ran}(q) \) and the restr to \( \text{Ran}(p) \)

and its restriction to \( \text{Ran}(q) \) is an iso.)

Lemma: A separable \( \Rightarrow V(A) \) and \( K_0(A) \) are stable.

Ex: Enough to show \( V(A) \) stable. If not, then \( \exists n \) and an able family of

\( x \in X \), \( \{ c_i \} \) s.t. \( \sup \| c_i \| > 1 \) when \( i \neq j \). So \( M_n(A) \) can be separable, so \( A \)

will either.

Ex 4: Let \( R \in L(E) \) be a factor of type II. Then \( \text{V}(R) \cong [0, \infty) \), via

for \( \psi \), \( \text{Ran}(p) \to \text{Ran}(q) \) the unique triv state, \( p \mapsto \text{V}(R) \).

So \( H(R) \cong R \) with \( (p \otimes q) \mapsto \text{V}(R) \).

Hence \( H(R) + = [0, \infty) \).