Let $R$ be a factor of type $II_1$, then $V(R) = [0, \omega]$ obvious setup question.

A factor $v$ is "simple VN alg." simple means no non-trivial weak up closed ideals. $M \subset L^1(\mathbb{H})$ unitarily contained VN alg. $M = v$, factor iff $M \cap M' = 0, 1$.

The set of closed ideals in $M$ is a set of exactly the same. This is the center of $M$, $Z(M)$.

For projections $p \in Z(M)$, $(C_p \mathbb{H})_p$ of $M = Z(M) \subset L^1(\mathbb{H}) \subset L^2(\mathbb{H}, \mathbb{I})$.

The simple reps (factors) can be divided into types.

Type I: $M = L^1(\mathbb{H})$, for some Hilbert space $\mathbb{H}$

- $\Gamma_\mathbb{H} = \text{dim}(\mathbb{H}) = n$, $M = M_n(C)$.
- $\Gamma_\mathbb{H} = H = C^2$.

Note: $L^1(\mathbb{H})$ is not simple as a C$^*$-alg., but is simple as a VN alg., since $K(\mathbb{H})$ is not dense in $L^1(\mathbb{H})$.

Type II: If $M$ has a tracial state, say $\tau$.

In fact, (ii) $\tau$ is unique.

To give a norm (via

To give a norm (via $M_* (C(M))$)

$V(M) = [0, \omega]$.

Example. Let $\Omega$ be any NHF alg. with tracial state $\tau$. Let $\pi_\Omega$ be the GNS repn assoc. to $\tau$. Then $\pi_\Omega'' = M$. If $x_0$ is std. cyclic vector from GNS, then

$U_n = (a_{n-1}, a_n)$ for $a \in M$ is the (unique) tracial state. $C^*$ is the closure in $B(\mathbb{H})$ of the closed reg.

Ex. Let $A \subset L^1(F_2)$, $(\Gamma_2, m$ two generators). $A''$ is a factor of type $II_1$.

Then (old) $\pi_\Omega''(\Omega)'''$ is the same for all NHF algns $\Omega$.

Then $\pi_\Omega''(\Omega)'''$ is the same for all NHF algns $\Omega$.

Open question: If we use $F_3$ in place of $F_2$, does one get a different VN alg.

For any discrete grp $G$, let $g \rightarrow g$ be the left reg repn of $G$ on $L^2(G)$, $(x,y)(h) = x(g^{-1}h)$ for $x \in L^2(G), y \in G$. Take $C^*_r(G)$ to be spann $\{x \ast y, g \in G\}$, reduced C$^*$-alg. of $G$.

Then $A$ above in $C^*_r(F_2)$ in its "natural" repn.
Open problem: If $m,n \geq 2, m + n \geq 2$, then $C^*_r(\mathbb{F}_p^{(m)}) \neq C^*_r(\mathbb{F}_p^{(n)})$. [All are known to have the same state, but we don’t know what it is.]

The (Pincher, Vazquez): $m \neq n \Rightarrow C^*_r(\mathbb{F}_p^{(m)}) \neq C^*_r(\mathbb{F}_p^{(n)})$.

Proof method: $K_0(C^*_r(\mathbb{F}_p^{(n)})) \cong \mathbb{Z}^n$.

“Def.” A state is type II if it has a tracial state and is not finite dimensional.

Proof of type II: if it has a tracial state and is not finite dimensional, then $M_\mathbb{N}$.

Good def of type II: it has a minimal nonzero projection for $\mathbb{N}$. Type III: $\mathbb{N}$ alg. completed tensor product $M \otimes L(\mathbb{N})$ for $M$ type II, and $L(\mathbb{N})$ a Hilbert space. But the possible values on projs are $[0, 1]$ instead of $\mathbb{Z}_\geq 0$.

Type III: Any two nonzero projs are $M_\mathbb{N}$ equivalent.

Recall: $M(A) = G(V(A))$.

Ex. $A = C^*_r(\mathbb{K}^n)$. Then $M(A) = 0$. Reason: The $M(A) = 0$. Reason: If $A$ is not unital, then $M(A) = 0$. Since this is $C^*_r(\mathbb{K}^n, M_n(\mathbb{C}))$, and any proj. vanishing at $0$ must be zero.

Ex. Let $L = L(\mathbb{F}_2)$ be the Calkin alg. Approach: $L = L(\mathbb{F}_2)$, $K = K(\mathbb{F}_2)$.

Let $\pi: L \rightarrow K$ be the quotient map. Claim: $V(\pi) = \Phi$, which vanishes at $0$ for $\mathbb{C}$, and $\pi$ vanishes at $0$ for $\mathbb{K}$.

In particular, any two nonzero projs in $L$ are $M_\mathbb{N}$ equivalent.

This is what we will prove. Reason it is enough: We need for $M_{\mathbb{N}}(L)$ to be equal to $M_{\mathbb{N}}(K)$, $\pi$ for $M_{\mathbb{N}}(L)$. But $M_{\mathbb{N}}(L) \cong K$ for $M_{\mathbb{N}}(L)$ as Hilbert spaces, and $M_{\mathbb{N}}(K) \cong K$ as von Neumann algebras.

To prove $M_{\mathbb{N}}(L)$ at nonzero projs $p, q \in L$:

Claim: $\forall p \in Q$ proj. $\exists$ proj. $e \in L$ st. $\pi(e) = p$.

Not generally true for such maps of $C^*$-alg. This gives $sp(a) \in K$.

Proof of claim: Choose any self-adjoint $a \in L$ st. $\pi(a) = p$. Then $\pi(a^*a) = \pi(a)^2$ for $a^*a \in K$. Then $\pi(a \cdot a^*)$ is a proj. So $\exists \lambda \in (0, 1)$ st. $\lambda^2 \geq 1 - \pi(a \cdot a^*)$ Choose $\lambda \in (a \cdot a^*)$.

Then $\lambda \in sp(a)$, define (unit functional calms) $e = X_{(\lambda, \lambda)}(a)$. [Fin is unit in sp(a)]

We have $\pi(X_{(\lambda, \lambda)}(a)) = \pi(\pi(a)) = \pi(\pi(a^*a)) = 1$.

Claim is proved.

What need for $\lambda > 0 < \gamma < \lambda$, for $X_{(\lambda, \gamma)}(a) = 0$, then $X_{(\lambda, \gamma)}(a) = 1$.

$\forall \lambda \in sp(a)$ so $X_{(\lambda, \gamma)}(a)$ is unit in $sp(a)$. 


To prove the $M$-$N$ eq. statement, let $p, q \in G$ be nonzero pps. Lift to pps in $L$.

They have infinite rank, since $r$ finite rank $\Rightarrow \tau_i (r) = 0$. So $\exists e \in G$ st. $se = e, es = t$.

Then $t = \tau_i(s)$ gives $t^2 = p, t^2 = q$. $M$-$N$ eq. statement is proved.

Now $\mathcal{V}(L) = \mathcal{V}(G)$ and $G(\mathcal{V}(L)) = 0$.

Defn. A $C^*$-alg. Then $A$ has real rank zero if for all $a \in A$, $\forall e \in A$, st. $b$ has finite spectrum and $\| b - a \| < e$.

Thm. A $C^*$-alg. Then $A$ has real rank zero if $A$ has the exchange property.

Thm. Let $\varphi : A \to B$ be a surj hom of $C^*$-algs. Assume $\text{Ker}(\varphi)$ has real rank zero and $\varphi : \text{K}_0(A) \to \text{K}_0(B)$ is surj. Then $\varphi(p) \in B$ $\exists$ proj. $p_0 \in A$ st. $\varphi(p) = p_0$.

Thm. (much later) Let $\varphi : A \to B$ be a surj hom of (unital) rings. Assume $\text{Ker}(\varphi)$ has the exchange property and $\varphi : \text{K}_0(A) \to \text{K}_0(B)$ is surj. Then $\forall \text{element} f \in B$, $\exists$ element $e \in A$ st. $\varphi(e) = f$. 