Dense to prove: \( K_0(A) = \{ [1_n] - [I_n] \} \in H(A^+) \) : \( K_0(p) = 1_n \).

\[
\text{Let } M_n(A^+) \quad 1_n = 1_n \theta 0. \quad \epsilon M_n(A^+)
\]

We had gotten. \( \eta \in K_0(A) \) given, a had written it is

\[
[e_0] - [f_0] = [e_0] - [f_0] \quad \epsilon \quad f_0 = f_0 \oplus (1 - f_0) \quad f_0 \epsilon \text{EM}_n(A^+).
\]

New states here. Set \( s = \begin{pmatrix} f_0 & 1 - f_0 \\ 1 - f_0 & f_0 \end{pmatrix} \). Check: \( s^2 = 1_{2n} \) so \( s \) is invertible.

\[
\begin{pmatrix} f_0 & 1 - f_0 \\ 1 - f_0 & f_0 \end{pmatrix}
\]

such that \( 1_0, K_0(e_0) = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \) (Can take any if \( e_0 \) proj).

We have \( s : A^+ \ni \lambda \mapsto s \lambda \). Let \( t = (\text{id}_{M_{2n}} \otimes \sigma) (b) \in \text{inv}(M_{2n}(A^+)) \).

Define \( p = t \circ s \). Check that \( K(p) = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \), because \( K \circ \tau = \text{id} \).

Recall notation: \( 0 \to A \xrightarrow{1} A \xrightarrow{p} B \to 0 \). Assume \( p \to A \) is the inclusion.

Lem: \( \text{inv}(B) \to \text{inv}(A) \) is the zero map. \( \Box \) Otherwise: \( \text{inv}(p) = 0 \).

Pf: Let \( \eta \in \text{inv}(B) \). Write \( \eta = [p] - [I_n] \) as above.

\[
\eta, 1_n \in \text{EM}_n(\mathbb{Z}_+). \quad \text{Def: } \tau \text{ induces } \pi^+ : A^+ \to B^+.
\]

\[
\tau_1(\eta) = C = \frac{\eta}{\text{vol}(A^+)} = \lambda \cdot 1_B^+.
\]

\[
K_0(p) = 1_n \oplus 0. \quad \text{So } \tau_1(p) = 1_n \oplus 0 = 1_n \oplus 0 = \frac{1_n \oplus 0}{\text{vol}(B^+)}.
\]

Thus \( \tau_1(p) = \pi_1(1_n) \) so certainly equivalent in \( H(A^+) \). Thus \( \pi_1(B(p)) = 0 \).

Now need lemma on \( \text{inv}(B(A)) \).

Notation: \( A \) is the ring of \( \epsilon_0(A) \) the set of invertible \( \epsilon \) in \( A \).

\( \epsilon \) in \( A \) is the convex component of \( \text{inv}(A) \) containing \( 1 \).

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\( \text{inv}(A(A)) \) is the convex component of \( \text{inv}(A) \) containing \( 1 \).

[Note: \( V(A) \) etc. need a topology here]
Lemma. A subset B on a. Then \( \text{inv}(A) = \frac{1}{n} \exp(a) \exp(-a) \exp(a) \exp(-a) \exp(a) \). 

**C-version:** A subset \( C \) of \( \mathbb{R} \). Then \( \exp(C) = \int \exp(a) \exp(-a) \exp(a) \exp(-a) \exp(a) \). 

This shows \( \text{inv}(A), \) \( \text{inv}(B) \) are pull, and both have \( x \rightarrow \exp(ax) \) a set path of invertible functions from \( I \) to \( \exp(a) \). Do this to all of the factorials.

**Generally:** \( \exp(A+B) \neq \exp(A) \exp(B) \). Do get equality if \( AB = BA \).

**Exercise:** prove equality if \( AB = BA \), and give counter-example in \( \mathbb{M}_2(C) \) otherwise.

**Proof.** Let \( G \) be the RHS. Claim: \( G \) is a subgroup. Certainly inverse.

**Given:** \( \exp(a)^{-1} = \exp(-a) \), so \( \exp(a)^{-1} = \exp(-a) \exp(-a) \exp(a) \exp(a) \exp(a) \exp(-a) \exp(-a) \). 

**Claim:** multiply in order of indices [not usually accepted]

**Claim:** \( G \) is open. 

Let \( s \in G \). Set \( \varepsilon = \| s - 1 \|^{-1} \). Suppose \( t \in A \) and \( \| s - t \| < \varepsilon \).

Then \( \| s^{-1} t^{-1} \| < \| s - t \| < \varepsilon \). 

So \( \exp(b) = s^{-1} t^{-1} \). So \( t = \exp(a) \). 

Claim proved.

**Proof:** 

\( \log (s^{-1} t^{-1}) = \log (1 - (s^{-1} t^{-1})) = -\frac{1}{2} (s^{-1} t^{-1})^2 - \frac{1}{3} (s^{-1} t^{-1})^3 - \cdots \)

The series converges absolutely since \( \| s^{-1} t^{-1} \| < 1 \). Or we have to calculus for rest of \( \log \).

To finish, note \( G \) is an open connected group. Need to know that an open subgroup of a top group is closed. 

**Proof:** \( H \) top gp, \( H \subset G \) open subgroup. Den \( H \uparrow A = \bigcup H_a \) (union of all \( a \in H \)). 

Costs almost like \( H \), which is clearly open. Claim follows.

Now we know \( G \) is connected, so under \( \text{inv}(A) \), and is closed and open, so \( \text{inv}(A), G \).

**Exercise:** Prove the \( C \)-version. (pf is essentially the same)

**Note:** \( u \) unit, \( \| u - 1 \| < 1 \) \( \Rightarrow \log(u) \in C \). Ask [cheap: we count fnl calculus, 

\( \log - i \) log is real valued on \( \mathbb{R} \).] 

(\( \log - i \) log is real valued on \( \mathbb{R} \).) 

**Key consequence:** Lemma. Let \( \text{inv}(A) \) be a surjective hom of unital Banach algs. Then \( \text{inv}(A) \) is surjective. 

[For \( C \)-algs: \( \text{inv}(A) \Rightarrow \text{inv}(B) \) surjective,]
Not true that \( \text{inv}(A) \to \text{inv}(B) \) is surjective.

**Ex.** Let \( X = \mathbb{C}, |x| \leq 1 \), let \( Y = \mathbb{C}, |y| = 1 \).

Let \( \text{AC} \) be \( \mathbb{C}, |z| = 1 \). Therefore, let \( U \in \text{AC} \) be the unitary \( u(z) = z \).

If \( z \in A \) is invertible and \( \overline{z} \notin A \), then the limiting number of \( u \) about \( 0 \) would be \( 0 \), but \( u \) is really 1.

**Proof:**

Clearly that \( \text{inv}_0(\pi(A)) \subseteq \text{inv}_1(B) \).

For reverse, let \( t \in \text{inv}_0(B) \). Write \( t = \frac{1}{\overline{z}} \exp(b) \) with \( b_1, \ldots, b_n \in \mathbb{R} \).

Choose \( a_k \in A \) so \( \pi(a_k) = b_k \). Set \( s = \sum \exp(a_k) \in \text{inv}(A) \). Then \( \text{inv}(s) = 0 \). \( \square \)

**Lemma.**

Let \( B \) be a unital Banach algebra. Let \( s \in \text{inv}(B) \). Then \( \left( \begin{array}{cc} s & 0 \\ 0 & s^{-1} \end{array} \right) \in \text{inv}(M_2(B)) \).

**Proof.**

Define \( \tilde{z}_\lambda \in M_2(B) \), for \( \lambda \in \mathbb{C} \), by

\[
\tilde{z}_\lambda = \begin{pmatrix}
\cos(\lambda) & \sin(\lambda) \\
-\sin(\lambda) & \cos(\lambda)
\end{pmatrix}.
\]

Then \( \tilde{z}_\lambda \) is unitary (check) \( \tilde{z}_0 = 1 \) and \( \tilde{z}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

Regard \( s \) being in \( M_2(B) \) via \( \tilde{z} \mapsto B \) as \( \lambda \mapsto \tilde{z}_\lambda \).

Set \( \tilde{t}_\lambda = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \tilde{z}_\lambda^{-1} \). Then \( s_i \) given by \( i = 1 \) and \( \tilde{t}_{1/2} = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \). \( \square \)

**For hol. func. calculus, need a hol. function defined in a nbhd of \( \text{sp}(a) \).**

**For power series: need a series \( \sum_{n=0}^{\infty} a_n z^n \) with radius of convergence \( R \) s.t. \( R > |a| \).** (True fact: easy to see that \( R > |a| \) will do). Then \( f(a) = \sum_{n=0}^{\infty} a_n a^n \).

**Spectral radius.**

(1) If series version is defined, then \( f \) is hol. in a nbhd of \( \text{sp}(a) \).

(2) In this case, \( f(a) \) via hol. calculus is equal to \( \sum a_n a^n \).

(Proof: use Cauchy formula.)

The hol. version doesn't apply if, for example, \( \text{sp}(a) = \overline{1} \) and \( f \) is hol. in \( \mathbb{C}, \frac{1}{2} < |a| < 2 \), but does not have a hol. extension to \( |a| < 2 \).