Corrections from last time. 

Let's denote \( L(A) \) should be homotopy classes in \( Z(A) = \{ h : S^1 \to \text{inv}(M_n(A)^+) \} \) with \( h(z) = 1 \) for \( z \in \mathbb{S}^1 \).

Recall, \( U \subset Z(A) \) and \( Z_n(A) = \frac{1}{n} \text{inv}(M_n(A)^+) \setminus M_n(A) \).

Continuity means (here a for rest of pt at Bott periodicity): range \( Z_n(A) \) for some \( n \), and cut at \( h(z) \). \( S^1 \to M_n(A)^+ \).

Example: \( L(A) \equiv K_n(A) \) using the "obvious" map.

Read: \( c \in M_n(A) \) under the. Then \( h_c(\delta) = 2\pi + 1 - e \), giving \( h_c : S^1 \to \text{inv}(M_n(A)^+) \).

Correct defn of \( \beta_A \). Use follows: Given \( c \in M_n(A) \), if \( c \in M_n(A) \), take \( \beta_A : \{ \delta \to M_n(A)^+ \} \to \text{inv}(M_n(A)^+) \).

Class of \( \delta \to h(\delta)c(h(\delta))^{-1} \). Need to check it is \( \in Z_n(A) \). With \( c : A^+ \to A \) being multiplication map, and suppressing "under " + notation: \( n \) need:

\[
K_A(c, (c(h(\delta)c(h(\delta))^{-1}) = (\delta c(\delta) + 1 - c(\delta)) (\delta h(\delta) + 1 - h(\delta))^{-1}.
\]

This is 1 since \( c(\delta) = h(\delta) \).

Exercise: \( \beta_A \) is well defined and a q.p. hom. \( \text{inv}(M_n(A)^+) \to Z_n(A) \).

There are two products in \( Z(A) \), pairwise mult. and \( \oplus \). These will sum up to homotopy by arguments already seen. So addition in \( L(A) \) can be defined using either.

We reduced pt that \( \beta_A \) is a q.p. hom. to under case. In this case (exercise) rendre:

\[
Z_n(A) = \{ h : S^1 \to \text{inv}(M_n(A)^+) \} \text{ with } h(z) = \sum_{0 \leq j \leq n} \beta_A\).
\]

We want to successively replace \( Z_n(A) \) with:

\[
Z_0(A) = \{ h \in Z_n(A) : \exists N, a_0, \ldots, a_N \in \mathbb{C} \text{ with } h(z) = \sum_{0 \leq j \leq N} \beta_A\).
\]

(\( \beta_0(A) \) says this is multible \( 1 \leq \delta \leq 1 \).

\[
Z_1(A) = \{ \text{with } h(\delta) = \sum_{0 \leq j \leq n} \beta_{A_j} \text{ and } \beta_{A_j} \text{ poly in } \delta, \}.
\]

\[
Z_2(A) = \{ \text{range of } c \to h_c \} \text{, where } h_c(\delta) = 2\pi + 1 - e \in Z_n(A) \text{ (don't need to use } h_c(h(\delta))^{-1} \text{, Exercise: Check that this is a correct defn of } \beta_A \text{ and identification of } \}
\]

Use homotopy classes in \( M^+ \) of these,

getting \( L^0(A) \) unnamed as rest. They are sums up to homotopy with \( \Theta \), and for \( Z_0(A), Z_1(A) \) equivalent (up to homotopy) with pairwise mult.
There will be four “redaction steps,” using four quite different methods.

**Lemma Z_n^0(A) \to Z_n(A) is a bijection on the set of homotopy classes, for each n.**

*Proof of Lemma* Main part: density of Z_n^0(A) in Z_n(A).

Claim 1. Bounded space, U \subset \mathbb{B} open. Then the Laurent poly fung S^1 \to U creates norm above in cod.

A fun S^1 \to U \leftarrow all thru C(S^1, U). (abuse of notation)

The lemma is not true if try to use unitaries in place of invertibles, since U(M_n(A)) is not open.

For pf of claim: C(S^1, U) is open in C(S^1, B), so enough to consider U = B.

If B = C, standard. So true if B is finite dimensional. So enough to show that if ps of C(S^1, B) with finite dim. range are dense in C(S^1, B). Given h : [0, 1] \to B with h(0) = h(1), and

\[ h(t) = h(x_0), \quad \| k \| = \| h(x_0) \| \]

k has finite range, so claim follows.

**Surjectivity:** Let h \in Z_n^0(A). Choose \epsilon > 0 \; N_\epsilon(h) \subset Z_n(A). Choose k \in N_\epsilon(h) given by \epsilon-neighborhood.

Laurent poly. The k \in Z_n^0(A) and k is homotopic to h in Z_n(A) by a straight line path.

**Injectivity:** Suppose h_0, h_1 \in Z_n^0(A) and h_0 \sim h_1 \; (\sim : \text{homotopy, not by M-approx equal}) in Z_n(A).

The homotopy is a fun \lambda \mapsto h_\lambda, which is an eff of Z_n(C([0, 1], A)). Using claim in

\[ B = M_n(C([0, 1], A)), \quad \text{choose} \; \epsilon > 0 \; N_\epsilon(h) \subset Z_n(C([0, 1], A)) \] and (here is where claim is used) choose k \in Z_n^0(C([0, 1], A)) with \| k - h \| < \epsilon. Want to show h_0 \sim h_1 in Z_n^0(A)

Use : straight line path from h_0 to h_1. Then k is homotopy from k_0 to k_1, the straight line path

from h_0 to h_1. All parts are through Laurent poly fans. 

**Conclusion (less explicit above):** h \in Z_n^0(A) \implies h has an inverse up to homotopy in Z_n^0(A).

Now: K_n(1A) can be identified with gp of homotopy classes in Z_n^0(A).

**Lemma** Grothendieck gp of \[ \Sigma A \] is \[ \Sigma \mathbb{Z}_n \]: h \in Z_n^0(A) for some n, \; N \in \mathbb{Z}_{>0}.

\[ \mathbb{Z}_n(S) = \mathbb{Z}^1 : S^1 \to M_n(1A), \]

(So h, Z_n have same matrix size.)

**Idea:** Start representing \[ K_n(1A) \] for A unitd as \[ \frac{\mathbb{Z}}{[\mathbb{Z} - 1]} : e \text{ idempotent}, \; n \in \mathbb{Z}_{>0}. \]
Exercise: Show this implies the lemma.

Let the claim: $k^y$ is homotopic to some cell of $Z_n(A)$ in $Z_n(A)$, say $k_0$. Choose $m_0$ st. $Z_{m_0} k_0$ is a polynomial. Then $Z_{m_0} k_0 : h \sim_{m_0} Z_{m_0}$ in $Z_n(A)$. Multiply through by $Z_n$, for suitable $m$, to make it a poly. Take $N = m_0 + m$, and $k = Z_m k_0$. Do get $k \sim_{m_0} Z_n$ in $Z_n(A)$.

To get lemma from claim: $G$ (homotopy classes in $Z_n(A)$) is, w.r.t. normal, the set of differences $[h_0] - [h_1]$ with $n \in \mathbb{Z}^+$, $h_0, h_1 \in Z_n(A)$. Want to take $h_1 = Z_n$ for some $N$. Choose $k, N$ as in the claim for $h_1 : k h_1 = L_{z_n}$. Then $[h_0] - [h_1] = L_{h_0} k \sim_{m_0} Z_{m_0}$.