We get to $K_p(S^A) = K_p(A)$ naturally.

Corollary. If $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ is exact, then there is a six term exact seq:

$$
\begin{array}{ccc}
K_p(I) & \rightarrow & K_0(A) \rightarrow K_p(C) \\
& \downarrow & \downarrow \\
K_1(B) & \rightarrow & K_1(A) \rightarrow K_1(I)
\end{array}
$$

("long exact seq" in alg top language).

Proof. Already have exact seq $\xymatrix{K_1(I) \rightarrow K_1(A) \rightarrow K_1(B) \rightarrow K_0(B) \rightarrow K_0(A) \rightarrow K_0(I)}$.

Fill up dotted arrow as $K_0(B) \rightarrow K_1(SB) \rightarrow K_0(SB) \rightarrow K_1(I)$.

Exercise. Give a description of the dotted arrow. When $A$ is small, and to simplify notation consider simplicial $c e B$, but $\mathbf{M_n}(SB)$. Choose any $a \in A$ s.t. $k(a) = e$.

From $v, \exp(2 \pi i \delta) \in A$, this is reasonable. Claim that $e$ is in $I^+ = I + IC_A$. To see this, $k(e) = \exp(2 \pi i \delta k(a)) = \exp(2 \pi i c e) = 1$. Claim in follows.

Direct limits. These make sense for systems $((A_v)_{v \in V}, (\phi_{ij})_{v \in V, j \in V})$ for a directed set $V$.

The order $\leq$ is supposed to be a partial order, directed.

Some condition is wanted for the index set, for a not.

What we have so far does not quite do.

For $B$ again, also need something like $|| \phi_{ij} || \leq 1$ $\forall i, j$ st $i \leq j$.

This is slight overkill, but must have:

For $V \in \mathcal{A}$, $\lim \phi_{ij}(v)$ exists. (Can't be $\infty$ and can't oscillate)

This is then $|| \phi_{ij} ||_{\mathcal{A}}$.

Prop. Let $I$ be a directed set and for $i \in I$, suppose we have exact seqs $0 \rightarrow J_i \rightarrow A_i \rightarrow B_i \rightarrow 0$ s.t. (shall now dispays) $(J_i), (A_i), (B_i)$ are direct systems, with commute, and diagrams $0 \rightarrow J_i \rightarrow A_i \rightarrow B_i \rightarrow 0$ commute.

Then $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ is exact.

Proof. Exercise. (Easier in C* case)
Lemma. Using notation above: \( (A - \log A) \), \( M_n(A) \equiv \log M_n(A) \) (natural identification).

\( \text{Pf. Exercise.} \)

Lemma. \( \text{For } x \in K_0 \text{ get } \frac{1}{2} C_0(x, A) = \log C_0(x, A) \) (natural identification).

[See for Banach alg's.]

\( \text{Pf. Exercise.} \) (For Banach alg case, use partitions of unity.)

Remark. \( A \) direct limit of simple \( C^* \)-alg's is simple. (One can see.)

Recall method. Maps must be injective. So consider \( A_c = C(A) \) with \( A_c \subseteq A \), when \( i \in \mathbb{N} \), and \( A \), closed. Suppose \( J \) is a ideal in \( A_c \) not \( M \text{ of } A_c \). Consider maps \( \frac{1}{2} C_0(x, A) \rightarrow \frac{1}{2} C_0(x, A/J) \).

They are not all zero, so some can be zero. \( A \) simple, \( C^* \)-alg.

They are non-trivial. By continuity \( A \rightarrow A/J \) is injective, \( J = 0 \).

One can say even: if \( J < A \rightarrow A/J \), then \( J = \log C_0(x, A) \).

Got single \( C^* \)-alg this way.

For Banach alg's, any \( C^* \)-alg's need not be \( K \)-_algebra. So clearly false. Problem: Find counterexamples. (Eg. with \( L^1 \) value alg's or even \( C^* \)-alg's \( L^1 \) in alg). \( \mathbb{K} \) and \( \mathbb{K} \) commute with direct limit: \( \mathbb{K} \) (Dir. limit) = \( \lim \mathbb{K} \mathbb{K} (x, A) = \lim \mathbb{K} \mathbb{K} (x, A) \).

Also: \( \mathbb{K} (\log A) \equiv \lim \mathbb{K} (\log A) \).

Check that \( \mathbb{K} \) are \( \text{finite-dim} \).

Will prove for \( K_1 \) with \( J = \mathbb{Z} \); for Banach alg's \( \frac{1}{2} C^* \)-alg, injective maps \( \mathbb{K} \).

Part (b). Can assume \( A_c \subseteq A \), \( A \) closed. \( \cdots \) \( \text{Banach alg's} \).

Need to prove: \( (2) \) \( \text{if } x \in C_0(x, A) \) then \( \exists \lambda \in \mathbb{C} \) s.t. \( \lambda \cdot x \in C_0(x, A) \).

\( \text{For } (2.1): \) Recall from finite-dim alg's. \( \exists \lambda > 0 \text{ s.th. } \lambda \cdot x \in C^* \)-alg and be \( B \), satisfies \( \| \lambda \cdot x \| \leq \delta \).

\( \text{Idea of proof.} \) Say \( x \) and what \( \delta \) is supposed to be. \( \| \lambda \cdot x \| \leq \delta \).

\( \lambda = \sqrt{\delta} \) \( \sqrt{\delta} \cdot x \in C_0(x, A) \).

Next, observe that \( \mathbb{K} (\mathbb{K})_{\mathbb{K}} \) is dense in \( A_{\mathbb{K}} \).
Now, given $p \in M_n(A)$, we can choose $n$ and $e \in M_n(A)^{\otimes n}$ such that $\|a - p\| < \frac{\epsilon}{4}$ for every $a$.

For every $a$, we have

$$\|a^{\otimes n}\| < \frac{\epsilon}{4}.$$

Choose $n_0$ such that $\|a^{\otimes n_0}\| < \frac{\epsilon}{4}$. Let $n = \min(n_0, m)$, $c = f_{g_0} e M_n(A)$, and $\|c - s\| < \frac{\epsilon}{3}$. Now, $\|c^{\otimes n} - e\| < \\|c^{\otimes n} - s^{\otimes n}\| + \|s^{\otimes n} - e\| < \frac{\epsilon}{3}$. Similarly, $\|c^{\otimes n} + f\| < 1$. Use of $\|s\| < \frac{\epsilon}{3}$.

From $g = (c^{\otimes n})^{\otimes n}$, we have $c^{\otimes n} = g^{\otimes n}$, and check that $g = c_{b_0} s^{\otimes n} = e_{gg} f_{gg} f_{gg}$. Then, $\|c_{b_0} s^{\otimes n} - e_{gg} f_{gg} f_{gg}\| < \frac{\epsilon}{3} < 2$.

One would define $K_0(A, B)$ to be $K_0(B)$ when $A = B$. For $Y \subseteq X$, one would define $K_0(Y) = K_0(X)$ for $Y \subseteq X$. This makes sense when working with compact supports.

One can define a "noncommutative unitary group" $U_n^{nc}$ to be the unital $C^*$-algebra generated by $U_{1k}$ for $1 \leq i, k \leq n$, subject to relations:

$$U_{1k} U_{1j} = U_{2i} U_{2j}$$

for $1 \leq i, j, k \leq n$. This is $2n^2$ equations in $U_n^{nc}$.

One would define a "natural" isomorphism $\text{Hom}(U_n^{nc}, A) \cong U(M_n(A))$. This is $2n^2$ equations in $U_n^{nc}$.
There are maps \( U_{11}^{nc} \rightarrow U_{11}^{nc} \), \( U_{1k}^{(n+1)} \rightarrow \sum U_{jk}^{(n+1)} \), \( \delta_{jk} = 1 \) when \( n + 1 \), \( 0 \) otherwise.

Contrast this with \( C^{\cdot, dg} \) (limit of \( C^{\cdot, dgy} \)). \( U_{\cdot, \cdot}^{nc} \rightarrow \lim U_{\cdot, \cdot}^{nc} \).

One gets: \( A \text{ unital } \Rightarrow K_{1}(A) \cong \lim U_{\cdot, \cdot}^{nc} \text{ (kernel homotopy class)} \).

There is a similar thing (with trickery so as not to get \( V(A) \) the \( K_{0} \)).