Recall from last time: $T$ is the Toeplitz algebra, generated by the unilateral shift $s$. Universal reps $\mathcal{B}$ an isometry $\pi: T \to \mathcal{B}$ such that $\pi(s) = v$.

Exact seq: $0 \to K \xrightarrow{\pi} C(s^*) \to 0$ \quad and \quad $\pi(s) = z, \quad z \in C(s) \cap z(s) = \mathbb{C}$

Maps $\iota: \mathcal{B} \to T, \quad \chi: T \to C(s) = \mathbb{C}$, \quad $\chi(\lambda) = \lambda 1$. \quad So \quad $\chi \circ \pi = \iota$.

$T_0 \subseteq T$ is generated $K \otimes T$ and $T \otimes C(s^*)$. Exact seq: $0 \to K \otimes T \xrightarrow{\pi_0} T_0 \xrightarrow{\pi_0} C(s^*) \to 0$

$T_1$ is a pullback of $T_0$ and $T_0$; recall when needed.

First of two homotopy lemmas:

Lemma: There exists $\phi_0, \phi_1: T \to T_0$ st. $\phi_0(s) = s \otimes 1, \quad \phi_0 = [s(1 - e_0) + e_0] \otimes 1$, and there is a homotopy $\lambda \mapsto \phi_{1, \lambda}$ from $\phi_0$ to $\phi_1$ such that $\pi_0(\phi_{1, \lambda}(s)) = z$ for all $\lambda$.

This part used in the statement last time.

Lemma used in its proof and later: Let $B$ be a unital $C^*$-algebra, let $s_1, s_2, \ldots, s_n \in B$ be partial isometries with mutual orthogonal projections $p_1, p_2, \ldots, p_n$. Let $s = \sum_{i=1}^n s_i$.

1. If $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n q_i = 1$, then $s$ is unital.
2. If $\sum_{i=1}^n p_i = 1$ and $q_1, q_2, \ldots, q_n$ are mutually orthogonal, then $s$ is an isometry.

In the proof of the homotopy lemma, we had checked that the intended values of $\phi_0(s)$ and $\phi_1(s)$ are isometries.

Easy to see $\pi_0(\phi_0(s)) = \pi_0(\phi_0(s))$ are both $z$.

Set $W_1 = s(1 - e_0) + e_0 s + s e_0$. We checked that it is a self-adjoint unitary. Also easy to see $W_1$ is an isometry.

Consider $C^*$-subalgebra $C \subseteq T_0 : C = \{ x \in T_0 : \pi_0(x) \in C(s^*) \}$. Wsl: $W_1 \in C$ is a self-adjoint unitary there. So there is a path $\lambda \mapsto W_{1, \lambda}$ from $W_1$ to $W_1 = W_{1,1} \otimes 1$, with values in $C$.

Then $W_{1, \lambda} = \pi_0(W_{1, \lambda}) W_{1, \lambda} \in C$, gives a homotopy path in $C$ from $0$ to $W_1$ such that

$\pi_0(W_{1, \lambda}) = 1$ for all $\lambda$. Then $C_{1, \lambda}$ is given by $C_{1, \lambda}(s) = W_{1, \lambda} s W_{1, \lambda}$. Certainly in $C$, $\pi_0(C_{1, \lambda}(s)) = \pi_0(W_{1, \lambda}) \pi_0(s) \pi_0(W_{1, \lambda})^{-1} = 1 - z$, as needed. Check $\phi_0 = \psi_0$ are a given.

$\phi_0(s) = s \otimes 1$, certainly $\pi_0(\phi_0(s))$ is supposed to be $s(1 - e_0) + e_0$, etc. Need to check $\pi_0(s(1 - e_0) + e_0) \otimes 1$. Need to check $\pi_0(W_{1, \lambda})$.

Lemma: There exists $\psi: T \to T$ st. $\psi(s) = s(1 - e_0) + e_0 s$, and homotopy $\lambda \mapsto \psi_{1, \lambda}$ from $\psi_0 = \psi$, $\psi_1 = \psi$, and $\pi_0(\psi_{1, \lambda}(s)) = z$ for all $\lambda$.

Check that $s(1 - e_0) + e_0 s$ is an isometry by using mutual orthogonal projections.
Set \( u_0 = 1 \), \( u_1 = 5(1-e)S + e_0S + e_0S + e_0S + e_0S \). Clearly, solving \( \lambda + \theta = 1 \): Look at initial and final parts:

Initial: \( (1-e,0) \) 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100

Final: \( (1-e,0) \) 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100

Also, \( \pi_0(u_0) = 1 \) (using \( \pi_j(S^2 \otimes 1) = 0 \), \( \pi_j(S^2 \otimes 1) = 1 \)). Clearly, \( \pi_0(u_0) = 1 \). So, with \( C \) as before, \( c \alpha_0 \) cannot map to \( u_1 \) using a unitary path in \( C \) by some argument as before. Can choose the path \( \alpha_0 \) to have \( \pi_0(\alpha_0) = 1 \) for all \( \alpha_0 \). Take \( \sigma_\alpha \) to be determined by \( \pi_0(u_0) = 1 \). Clearly, \( \pi_0(c\alpha_0) = \pi_0(\alpha_0) \pi_0(c\alpha_0) = 1 \). \( \alpha_0 \) is zero since \( \pi_0(\alpha_0) = 1 \). For other, use \( S^2 = 1 \), to get \( S(1-e) \otimes 1 \) and \( e_0 \otimes 1 \).

**Lemma:** Let \( \mathcal{C} \) be a factor from \( A \) to \( A \) satisfying \( \alpha_0(\alpha_0) = 1 \) holding invariance, middle exactness, and stability. Then \( A \in \mathcal{C} \). Let \( E : E(\triangle) \to E(\triangle) \) be defined by \( E(\triangle) \). If \( \triangle \) is a \( A \)-operator, then \( E(\triangle) \). For a \( A \)-operator, \( \alpha_0 \) as appropriate. All \( A \) are normal, so this preserves exact seqs.

Recall \( T_0 = \{ (x,y) \in T_0 \otimes T_0 : E(x,y) = x \} \), with \( T_0 = E(\triangle) \). Define \( T_0 \) on \( T_0 \). \( \alpha_0 \) is zero.

**Claim:** If \( \alpha_0 \) holds, \( \alpha_0 : T_0 \to T_1 \) is a homomorphism.

\( \omega(\alpha_0) = \{ (x,y) \in T_0 \otimes T_0 : x \in K \otimes T_0, y \in T_0 \} \), \( \omega(\alpha_0) = 1 \).

Need to check: All three supposed formulas satisfy \( \pi_0(u_0) = \pi_0(u_1) \), so \( \pi_0(u_0) \).

Also, by considering initial and final parts, \( (S(1-e) \otimes 1, S) \) is an isometry \( \pi_0(u_0) \). Already know \( \pi_0(u_0) \).

Let \( p_0 : T \to T_0 \) be \( p_0(x) = (x \otimes 1, 0) \), and let \( p = \pi_0 \). Observe \( \pi(1 + \theta) \subset A \), \( \pi(1 + \theta) \subset T_1 \), so \( \alpha_0 + \theta \) and \( \alpha_0 + \theta \) make sense as maps \( T \to T_1 \), and \( E(\alpha_0 + \theta) = E(\alpha_0) + E(\theta) \).

Claim: \( \alpha_0 + \theta = \alpha_0 + \theta \). As \( \alpha_0 \).

For \( \alpha_0 \): \( \alpha_0 = (x,y) \in T_0 \otimes T_0 \), so \( \alpha_0 = (x,y) \in T_0 \otimes T_0 \). Also \( \pi_0(u_0) = \pi_0(u_1) \), so \( \pi_0(u_0) = \pi_0(u_1) \). The case \( \alpha_0 \).
For $\lambda > 0$, we have $\lambda \mapsto \gamma_\lambda$ from first homotopy lemma to get $\gamma \mapsto (\gamma_\lambda(s), \lambda s)$. Extened to homotopy if maps $T \to T$ (Note $\pi_0(y \times I, s) \cong \pi_0(y)$, so we really are in $T_1^\lambda$. Check that at $\lambda = 0$ we get $\gamma_0(s)$.

In particular, and at $\lambda = 1$ get $\gamma_1(s)$ (go back and check). Claims are done.

Now, $E(\alpha) + E(\rho \circ \alpha) = E(\alpha + \rho \circ \alpha) = E(\alpha) = E(\alpha + \rho) = E(\alpha) + E(\rho)$, so

$\overline{E(\alpha) + E(\rho)} = E(\alpha) + E(\rho)$.

$E(\rho)$ is n.h.i. by split exactness of $0 \to \mathcal{N} \to T \to T \to 0$ and $E(\rho)$ is an isomorphism by matrix stability. So $E(\rho) \circ E(\gamma) = id$.

Consider: direct system $C(S^n) \to M_2(C(S^n)) \to M_2(C(S^n)) \to \cdots$, limit $A_\infty$.

\[
 \begin{array}{c}
 f \mapsto \text{diag}(f, f(x_0)) \\
 \text{constant for } C(S^n, M_2) \end{array}
\]

\[
 \text{const for } \gamma_0 \in C(S^n, M_2).
\]

Constraint: $\forall n, \gamma^n \mapsto \gamma_\infty$ is dense in $S^n$. (all $\gamma^n$ is dense)

One can calculate $K_n(A_\infty)$. For $n$ even, get $K_0(A_\infty) = \mathbb{Z} \oplus \mathbb{Z}/2$, $K_1 = 0$

For odd, $K_0 = \mathbb{Z} [\frac{1}{2}]$, $K_1(A_\infty) = \mathbb{Z}$.

Generators of copies of $\mathbb{Z}$ "come from" the odd genera of $H^n(S^n; \mathbb{Z})$ for $n > 0$.

So $A_\infty \not\cong A_1$. In fact, even, odd, $n$ odd $\Rightarrow A_m \cong A_n$ since they have different $K$-theory.

What about $m, n$ distinct but both even? One might hope to distinguish $A_2$ and $A_4$ using a $\mathbb{Z}$-graded cohomology theory for $C^*$-algebras.

This is hopeless. In fact, $A_2 \cong A_4$. Generally, $m, n$ both even $\Rightarrow A_m \cong A_n$

$m, n$ both odd $\Rightarrow A_m \not\cong A_n$.