EQUIVARIANT K-THEORY (MATH 691; PHILLIPS; WINTER 2020)

N. CHRISTOPHER PHILLIPS

This is an initial version of the course description for N. C. Phillips’ Math 691, on equivariant K-theory of topological spaces, at the University of Oregon, Winter Quarter 2020. It is subject to revision as decisions get made through the quarter on which of the possible topics to discuss and which to omit.

Here is what I wrote as the course proposal.

This is a proposal for a topics course on equivariant K-theory for actions of compact groups on compact spaces. The course will assume knowledge of topological K-theory, defined using finite dimensional vector bundles. It is intended to be accessible to both topologists and people working on Banach algebras, so Banach algebra material will be mentioned but will not be needed for most of the course.

List of topics. Throughout, \( G \) is a compact group, and the theory is already interesting when the group is finite.

- The representation ring of \( G \). (This is the coefficient ring for \( G \)-equivariant K-theory.)
- Basics: homotopy invariance, exact sequences, etc. These are very similar to the nonequivariant case, and will be gone through fast.
- Equivariant Bott periodicity. I don’t expect to be able to prove the most general statement, but will describe it. (The only proof I know uses elliptic pseudodifferential operators.)
- The Atiyah-Segal Completion Theorem. When \( G \) acts on \( X \), this theorem identifies the nonequivariant (representable) K-theory of \( X \times_G EG \) with a particular completion of the \( G \)-equivariant K-theory of \( X \). (Equivariant cohomology is sometimes defined to by the cohomology of \( X \times_G EG \). It follows from this theorem that equivariant K-theory contains more information than that definition would suggest.)
- Further topics, possibly including some discussion of the Künneth formula for the case when \( G \) is cyclic of prime order (what to do for more general finite groups is an open problem) and the background for the fact that equivariant K-theory is the same as the ordinary K-theory of the crossed product algebra.

The most common reference for ordinary (nonequivariant) topological K-theory seems to be Atiyah’s book [1]. There are some comments in it about equivariant K-theory, and equivariant K-theory is actually used in some computations of ordinary

\[ \text{Date: 4 January 2019.} \]
K-theory in the later part of the book. Otherwise, as far as I know, equivariant K-theory isn’t in textbooks, except that the C* version is in Chapter 11 of [4] (which is based on Chapter 2 of [6]).

The original reference for equivariant K-theory of compact spaces is [9]; the predecessor paper [8] is on the representation ring of a compact group, and has much more than we will need.

The general form of equivariant Bott periodicity is in [2]. There are indications in [1] of how to get a weaker but still often useful statement without using elliptic pseudodifferential operators.

The Atiyah-Segal Completion Theorem is in [3]. (There is a version for C*-algebras, in [7].)

The material related to the equivariant Künneth formula for finite cyclic groups of prime order is in [5]. As far as I know, there is no corresponding published paper; Koehler left mathematics after his Ph.D. There is actually no mention of equivariant Künneth formula in his thesis; instead, it is about what C*-algebraists call the Universal Coefficient Theorem. Caution: this is not the same as what one might think based on the usual Universal Coefficient Theorem in algebraic topology, although there is a formal resemblance. If you know how to prove the equivariant Universal Coefficient Theorem, then you know everything you need in order to prove the equivariant Künneth formula, although, as far as I know, nobody has written this down. The formalism in [5] relies heavily on the equivariant version of Kasparov’s two variable generalization of the K-theory of C*-algebras. I know where to tell somebody to start if they want to do this in ordinary algebraic topology, but I don’t know whether it can reasonably be done.

References