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The inverse problem for primitive ideal spaces

Hergen Harnisch and Eberhard Kirchberg

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Abstract

A pure topological characterization of primitive ideal spaces of separable nuclear C*-algebras is given. We show that a T_0-space X is a primitive ideal space of a separable nuclear C*-algebra A if and only if X is point-complete (cf. Definition A.1), second countable and there is a continuous pseudo-open and pseudo-epimorphic map (Definition 1.3) from a locally compact Polish space P into X. We use this pseudo-open map to construct a Hilbert bi-module $\mathcal{H}$ over $C_0(P, \mathbb{K})$ such that X is isomorphic to the primitive ideal space of the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ generated by $\mathcal{H}$. Moreover, our $\mathcal{O}_{\mathcal{H}}$ is $\mathrm{KK}(X; ., .)$–equivalent to $C_0(P)$ (with action of X on $C_0(P)$ given be the natural map from $\mathbb{O}(X)$ into $\mathbb{O}(P) \cong \mathbb{I}(C_0(P))$, and in the sense of [14, sec. 4]). Our construction becomes almost functorial in X if we tensor $\mathcal{O}_{\mathcal{H}}$ with the Cuntz algebra $\mathcal{O}_2$.

Contents

1 Introduction and main results

Recall that for every separable C*-algebra A its space of primitive ideals $\mathrm{Prim}\,(A)$ with the Jacobson hk-topology is a second countable, locally quasi-compact T_0-space which is a continuous and open image of the Polish space $P(A)$ of pure states on A. There is a well-known lattice isomorphism from the lattice $\mathbb{I}\,(A)$ of closed ideals of A onto the lattice $\mathbb{O}\,(\mathrm{Prim}\,(A))$ of open subsets of $\mathrm{Prim}\,(A)$ given by $I \in \mathbb{I}\,(A) \mapsto U_I := \{J \in \mathrm{Prim}\,(A) : \ I \not\subset J\}$ (the support of I in $\mathrm{Prim}\,(A)$). The correspondence maps closed linear spans (respectively intersections) of families of closed ideals of A to unions (respectively interiors of intersections) of the corresponding open subsets of $\mathrm{Prim}\,(A)$. We will use this isomorphism and its properties frequently without mentioning it.

We call a point-complete (Def. A.1), second countable, locally quasi-compact T_0-space a *Dini space*, because its structure is determined completely by its Dini functions (cf. Subsection A.1, Section 6 and [16] for our notations on T_0-spaces). For example, $\mathrm{Prim}\,(A)$ is point-complete for separable A, because it is the continuous and open image of $P(A)$. Point-complete second countable T_0-spaces with linearly ordered lattice of open subsets are examples of Dini spaces. One has moreover that every Dini space is the image of a Polish space by an open and continuous map (cf. [18]). It is an unsolved problem whether *all* Dini spaces are primitive ideal spaces of separable C*-algebras or not. Here we limit ourselves to the characterization of the primitive ideal spaces of separable *nuclear* C*-algebras. We get a complete description in pure topological terms with help of a result of [20], see below.

We suppose that the reader is not very familiar with our topological terminology for T_0-spaces and its maps. We explain some of our later used topological ideas with help of open continuous epimorphisms (instead of the later used pseudo-open pseudo-epimorphisms).

Consider for example a T_0-space X that is the image of a Polish space P under a continuous and open map π (as it happens in the case of a Dini space X):

One can consider the map $\Psi(U) := \pi^{-1}U$ from the lattice $\mathbb{O}(X)$ of open subsets U of X into the lattice of open subsets of P. Then Ψ is a lattice monomorphism from $\mathbb{O}(X)$ into $\mathbb{O}(P)$ with $\Psi(X) = P$ and $\Psi(\emptyset) = \emptyset$ that preserves l.u.b. and g.l.b., i.e. satisfies the following properties (I)–(IV).

Definition 1.1. A map Ψ from the lattice $\mathbb{O}(X)$ of open subsets of a T_0-space into the lattice $\mathbb{O}(P)$ of a T_0-space P is a *l.u.b.- and g.l.b.-preserving lattice monomorphism* if it has the following properties (I)–(IV):

(I) $\Psi^{-1}(P) = \{X\}$ and $\Psi(\emptyset) = \emptyset$.

(II) $\Psi\left((\bigcap_\alpha U_\alpha)^\circ\right) = \left(\bigcap_\alpha \Psi(U_\alpha)\right)^\circ$ for every family of open subsets $U_\alpha \subset X$. (Here Z° denotes the interior of a subset Z of X respectively P.)

(III) $\Psi\left((\bigcup_\alpha U_\alpha)\right) = \bigcup_\alpha \Psi(U_\alpha)$ for every family of open subsets $U_\alpha \subset X$.

(IV) $\Psi(U_1) = \Psi(U_2)$ implies $U_1 = U_2$ for all $U_1, U_2 \in \mathbb{O}(X)$.

(I)–(IV) means equivalently that $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(P)$ defines a lattice isomorphism of the lattice $\mathbb{O}(X)$ of open subsets of X onto a sub-lattice $\mathcal{Z}$ of $\mathbb{O}(P)$ that contains P and $\emptyset$ and is closed under l.u.b. (unions) and g.l.b. (interiors of intersections).

Definition 1.2. A C*-subalgebra C of a C*-algebra E is called *regular* in E, if

(i) C separates the closed ideals of E (i.e. $I \cap C = J \cap C$ implies $I = J$ for every $I, J \in \mathbb{I}(E)$),

(ii) $(I + J) \cap C = (I \cap C) + (J \cap C)$ holds for every $I, J \in \mathbb{I}(E)$.

If $C \subset E$ is regular in E, then the map $\Psi \colon \mathbb{O}(\mathrm{Prim}(E)) \to \mathbb{O}(\mathrm{Prim}(C))$ that corresponds to the map $J \mapsto C \cap J$ from $\mathbb{I}(E) \cong \mathbb{O}(\mathrm{Prim}(E))$ into $\mathbb{I}(C) \cong \mathbb{O}(\mathrm{Prim}(C))$ satisfies (I)–(IV) of Definition 1.1 (cf. Lemma A.14).

In the case where X is Hausdorff, (I)–(IV) of Definition 1.1 conversely imply that π is an open epimorphism. But since in our case X is only a T_0-space, the properties (I)–(IV) do not imply that π is an *open* map onto X (see Example A.13). In fact, in the case of a point-complete T_0-space X one has that for maps $\Psi \colon \mathbb{O}(X) \to \mathbb{O}(P)$ with properties (I)–(IV) there is a unique map $\pi \colon P \to X$ with $\Psi(U) = \pi^{-1}U$ for open $U \subset X$ and this (continuous) map π is pseudo-open and pseudo-epimorphic in the sense of the following definition (cf. Proposition A.11).

Definition 1.3. Suppose that X and P are T_0-spaces. We define the *pseudo-graph* R_π of a continuous map $\pi \colon P \to X$ by

$$R_\pi := \left\{ (p,q) \in P \times P : \;\; \pi(q) \in \overline{\{\pi(p)\}} \right\}. \tag{1.1}$$

A subset Z of P is R_π-invariant[1] if $q \in Z$ and $(p,q) \in R_\pi$ imply $p \in Z$. We call the map $\pi \colon P \to X$

(i) *pseudo-open* if the natural map $(p,q) \in R_\pi \mapsto p \in P$ is an open map from the space R_π into P, and the image $\pi(V)$ of every R_π-invariant open subset V of P is an open subset of $\pi(P)$,

(ii) *pseudo-epimorphic* if $\pi(P) \cap F$ is dense in F for every closed subset F of X (i.e. if $U \setminus V$ contains a point of $\pi(P)$ for every pair $V \subset U$ of open subsets of X with $V \neq U$).

R_π is the ordinary equivalence relation on P defined by π if X is a T_1-space. Note that pseudo-open (respectively pseudo-epimorphic) maps $\pi \colon P \to X$ are open (respectively epimorphic) if X is a T_1-space. In general, R_π is not a closed subset of $P \times P$ (even if P is Polish).

We are now in position to state our main result:

Theorem 1.4. *Suppose that X is a* point-complete T_0*-space which has a faithful map Ψ from $\mathbb{O}(X)$ into the open sets $\mathbb{O}(P)$ of a locally compact Polish space P with the properties (I)–(IV) of Definition 1.1. Then there*

[1] R_π is a partial order on P. Being R_π-invariant is the same as "downward closed" or "lower invariant" in lattice-theoretic sense (where $p \leq_R q \iff (p,q) \in R$).

exists a separable inductive limit E of type I C^**-algebras and an automorphism σ of E such that X is homeomorphic to the primitive ideal space of the* C^**-algebra crossed product $E \rtimes_\sigma \mathbb{Z}$. Moreover, E and σ can be chosen such that:*

*(i) $E \rtimes_\sigma \mathbb{Z}$ is isomorphic to a Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H}(A,h))$ of a Hilbert A-bi-module $\mathcal{H}(A,h)$ given by a non-degenerate *-homomorphism $h\colon A \to \mathcal{M}(A)$ with $h(A) \cap A = \{0\}$ and $A \cong C_0(P) \otimes \mathbb{K}$.*

(ii) The closed ideals of $E \rtimes_\sigma \mathbb{Z}$ are in 1-1-correspondence with the σ-invariant closed ideals of E.

(iii) $A \cong C_0(P) \otimes \mathbb{K}$ is a regular C^**-subalgebra of $E \rtimes_\sigma \mathbb{Z}$, is contained in E, and E is the smallest σ-invariant* C^**-subalgebra of E containing A. The map Ψ is induced by*

$$J \in \mathbb{I}(E \rtimes_\sigma \mathbb{Z}) \mapsto J \cap A \in \mathbb{I}(A) \cong \mathbb{O}(P)$$

via the homeomorphism of $\mathrm{Prim}(E \rtimes_\sigma \mathbb{Z})$ *with X.*

A full hereditary C*-subalgebra C of E is always regular in E, because $J \mapsto C \cap J$ is a lattice isomorphism from $\mathbb{I}(E)$ onto $\mathbb{I}(C)$. Thus, by Theorem 1.4(iii), $C_0(P)$ is isomorphic to a regular Abelian C*-subalgebra of $E \rtimes \mathbb{Z} \cong A_X$. In [20] it has been shown that for every separable *nuclear* C*-algebra B there is an Abelian regular C*-subalgebra C of $B \otimes \mathcal{O}_2$ with (at most one-dimensional) maximal ideal space $P := \mathrm{Prim}(C)$, (cf. [20, thm. 6.11]).

Note that $\mathrm{Prim}(B) \cong \mathrm{Prim}(B \otimes \mathcal{O}_2) \cong \mathrm{Prim}(B \otimes \mathcal{O}_2 \otimes \mathbb{K})$. Thus, in conjunction with our solution of the inverse problem, Theorem 1.4(iii), and by Proposition A.11 the following characterization of primitive ideal spaces of separable nuclear C*-algebras:

Corollary 1.5. *A* T_0 *space X is isomorphic to the primitive ideal space of a separable nuclear* C^**-algebra, if and only if,*

(i) X is point-complete (cf. Definition A.1)[2]

(ii) there exists a locally compact Polish space P and a pseudo-open and pseudo-epimorphic continuous map $\pi\colon P \to X$ in the sense of Definition 1.3.

[2] "point-complete" is called "spectral" in [11] and "sober" by others

By a result of the second named author every isomorphism κ from the primitive ideal space $\mathrm{Prim}\,(B)$ onto the primitive ideal space $\mathrm{Prim}\,(A)$ of separable nuclear C*-algebras A and B can be realized by an isomorphism φ from $A\otimes\mathcal{O}_2\otimes\mathbb{K}$ onto $B\otimes\mathcal{O}_2\otimes\mathbb{K}$ with $\varphi(\kappa(J)) = J$ for $J \in \mathrm{Prim}\,(B \otimes \mathcal{O}_2 \otimes \mathbb{K}) \cong \mathrm{Prim}\,(B)$. Moreover, φ with this property is uniquely determined up to unitary homotopy by unitaries in the multiplier algebra of $B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ (cf. Definition 2.11 and [15, cor. L]). In particular, *there is up to isomorphisms exactly one separable stable nuclear* C*-*algebra* A *with* $\mathrm{Prim}\,(A) \cong X$ *and* $A \otimes \mathcal{O}_2 \cong A$.

More generally: Suppose that A and B are stable separable C*-algebras, where A is exact and B is strongly purely infinite (in the sense of [19, def. 5.1], e.g. that $B \otimes \mathcal{O}_\infty \cong B$ or even $B \otimes \mathcal{O}_2 \cong B$). Then every map Ψ from $\mathbb{O}\,(\mathrm{Prim}\,(B))$ to $\mathbb{O}\,(\mathrm{Prim}\,(A))$ that satisfies the properties

(I) $\Psi^{-1}(\mathrm{Prim}\,(A)) = \{\mathrm{Prim}\,(B)\}$ and $\Psi(\emptyset) = \emptyset$,

(II) $\Psi\left((\bigcap_\alpha U_\alpha)^\circ\right) = \left(\bigcap_\alpha \Psi\,(U_\alpha)\right)^\circ$ for every family $\{U_\alpha\}$ of open subsets of $\mathrm{Prim}\,(B)$,

(III_0) $\Psi\left((\bigcup_\alpha U_\alpha)\right) = \bigcup_\alpha \Psi\,(U_\alpha)$ for every *upward directed* net of open subsets $U_\alpha \subset \mathrm{Prim}\,(B)$,

can be realized by a *non-degenerate* nuclear *-monomorphism h from $A \otimes \mathcal{O}_2\otimes\mathbb{K}$ into $B\otimes\mathcal{O}_2\otimes\mathbb{K}$ with $\Psi(J) = h^{-1}(h(A)\cap J)$ for $J \in \mathbb{I}\,(B \otimes \mathcal{O}_2 \otimes \mathbb{K}) \cong \mathbb{O}\,(\mathrm{Prim}\,(B))$, and h is uniquely determined by this property up to unitary homotopy (cf. [15, thm. K] and Definition 2.11). (Note here that (III_0) does not imply $\Psi(U) \cup \Psi(V) = \Psi(U \cup V)$.)

This implies that our construction of a separable nuclear algebra A with given primitive ideal space X is (up to unitary homotopy) a contravariant functor on the maps $\Psi\colon \mathbb{O}\,(\mathrm{Prim}\,(B)) \to \mathbb{O}\,(\mathrm{Prim}\,(A))$ with (I), (II) and (III_0) if we tensor with $\mathcal{O}_2 \otimes \mathbb{K}$.

Theorem 1.4 also implies that a separable C*-algebra B has a primitive ideal space $X = \mathrm{Prim}\,(B)$ which is the continuous pseudo-open and pseudo-epimorphic image of a locally compact Polish space P if and only if $B \otimes \mathcal{O}_2$ contains a regular Abelian C*-subalgebra in the sense of Definition 1.2 (see Corollary 5.3).

Since $h(A) \cap A = \{0\}$ for our Hilbert bi-module $\mathcal{H}(A, h)$, we get from [24, cor. 3.14] that $\mathcal{T}(\mathcal{H}(A, h)) \cong \mathcal{O}\,(\mathcal{H}(A, h))$ and then (by [24, thm. 4.3]) that the natural embedding $A \hookrightarrow \mathcal{O}(\mathcal{H}(A,h))$ induces a KK-equivalence between $\mathcal{O}(\mathcal{H}(A,h))$ and A. Thus $E \rtimes_\sigma \mathbb{Z} \cong \mathcal{O}(\mathcal{H}(A,h))$ is KK-equivalent to $C_0(P)$, which may give sometimes different nuclear C*-algebras that have the same primitive ideal space X but have not the same K-groups. The natural embedding from A into $B := \mathcal{O}(\mathcal{H}(A,h))$ is Ψ-equivariant with respect to the natural actions Ψ_A and Ψ_B of $\mathbb{O}(X) \cong \mathbb{I}(B)$ on A and B, because it is compatible with the natural action of $\mathbb{O}(X)$ on $\mathcal{H}(A,h)$, $\mathcal{L}(\mathcal{H}(A,h))$, $\mathcal{F}(\mathcal{H}(A,h))$ and $\mathcal{T}(\mathcal{H}(A,h)) \cong B$.

Therefore it defines also an element of the group $\mathrm{KK}(X; A, B)$. A closer look to the proof of [24, thm. 4.4] shows that the Kasparov B-A-module $(\mathcal{F}(\mathcal{H}(A,h)) \oplus \mathcal{F}(\mathcal{H}(A,h)), \pi_0 \oplus \pi_1, T)$ of [24, def. 4.3] (which defines the KK-inverse $\beta \in \mathrm{KK}(B,A)$ of $[A \hookrightarrow B]$) carries in a natural way an action of $\mathbb{O}(X)$ such that the B-A-module becomes Ψ-equivariant and defines therefore an element of $\mathrm{KK}(X; B, A)$. The homotopy constructed in in the proof of [24, thm. 4.4] turns out to be also Ψ-equivariant and can be used to prove that *the natural inclusion $A \hookrightarrow \mathcal{O}(\mathcal{H}(A,h))$ defines a $KK(X; ., .)$–equivalence between A and $\mathcal{O}(\mathcal{H}(A,h))$.*

We can replace P by $\mathbb{R}_+ \times P$ in the proof of Theorem 1.4. Then the $E \rtimes_\sigma \mathbb{Z}$ becomes KK-trivial. It becomes even $\mathrm{KK}(X; ., .)$-trivial, and thus absorbs $\mathcal{O}_2$ tensorially by [15, chp. 1,cor. N] and the following observation.

The *-monomorphism $h\colon A \to \mathcal{M}(A)$ in our special construction of $\mathcal{H}(A,h)$ (from Ψ_A and $A = C_0(P) \otimes \mathbb{K}$) is unitarily equivalent to its infinite repeat $\delta_\infty \circ h$. Then $\mathcal{H}(A,h)$ is isomorphic to $\mathcal{H}(A,h) \otimes \mathcal{H}(\mathbb{K}, \delta_\infty)$. One can show that the latter implies that $\mathcal{O}(\mathcal{H}(A,h))$ is isomorphic to $\mathcal{O}(\mathcal{H}(A,h)) \otimes \mathcal{O}_\infty$. In particular, our algebras $E \rtimes \mathbb{Z}$ in Theorem 1.4 are strongly purely infinite.[3]

In general $\mathcal{O}(\mathcal{H}(A,h))$ need *not* be purely infinite if one works with the weaker assumptions on $\mathcal{H}(A,h)$ of Corollaries 4.26–4.32, as Rørdam's example of a stably infinite nuclear separable stable simple C*-algebra shows (which is KK-equivalent to $C(S^2 \times S^2 \times \cdots)$ by [24] and is *not* purely infinite, cf. [25]).

We recall in the Appendix some needed basic facts on T_0-spaces, pseudo-open maps, Hilbert C*-modules and crossed products by $\mathbb{Z}$. We give there elementary proofs for the needed results.

[3] Details on strong pure infiniteness of $\mathcal{O}(\mathcal{H}(A,h))$ and on the $\mathrm{KK}(X; ., .)$–equivalence of A and $\mathcal{O}(\mathcal{H}(A,h))$ will be published elsewhere.

2 Realization of ideal-lattice morphisms

2.1 Hilbert bi-modules and cones of c.p. maps

Definition 2.1. Suppose that A and B are C*-algebras, and let $CP(A,B)$ denote the cone of completely positive maps from A into B. A subset $\mathcal{C}$ of $CP(A,B)$ is an *operator convex cone* of c.p. maps if $\mathcal{C}$ has the following properties (i)–(iii).

(i) $\mathcal{C}$ is a cone.

(ii) If $V \in \mathcal{C}$ and $b \in B$, then the map $a \mapsto b^*V(a)b$ belongs to $\mathcal{C}$.

(iii) $V_{r,c}\colon a \in A \mapsto c^*V \otimes \mathrm{id}_n(r^*ar)c$ is in $\mathcal{C}$ for every $V \in \mathcal{C}$, every row-matrix $r \in M_{1,n}(A)$ and every column-matrix $c \in M_{n,1}(B)$.

$\mathcal{C}$ is *full* if the linear span of $\{V(a) : \ V \in \mathcal{C}, a \in A\}$ is dense in B, and $\mathcal{C}$ is *separating* if $V(a) = 0$ for all $V \in \mathcal{C}$ implies $a = 0$.

It is easy to see that the point-norm closure of an operator convex cone is again operator convex.

Let S be a subset of $CP(A,B)$. We denote by $K(S)$ the smallest subcone of $CP(A,B)$ which is invariant under the operations in (ii) and (iii), and by $\mathcal{C}(S)$ the point-norm closure of $K(S)$ (i.e. the closure of $K(S)$ in $\mathcal{L}(A,B)$ w.r.t. the strong operator topology). Then $K(S)$ and $\mathcal{C}(S)$ are operator convex cones of completely positive maps. We say that $S \subset \mathcal{C}$ *generates* the (point-norm closed) operator convex cone $\mathcal{C}$ if $K(S)$ is dense in $\mathcal{C}$. $\mathcal{C}$ is *countably generated* if a countable subset S of $\mathcal{C}$ generates $\mathcal{C}$.

Note that

$$V_{r,c}(a) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i^* V(r_i^* a r_j) c_j$$

for $(r_1,\ldots,r_n) = r \in M_{1,n}(A)$ and $(c_1,\ldots,c_n) = c^t \in M_{1,n}(B)$.

Remark 2.2. If E is a Hilbert B-module and $h\colon A \to \mathcal{L}(E)$ a *-homomorphism, then the Hilbert B-module sum $E_\infty := E \oplus E \oplus \cdots$ with right A-module structure given by $h_\infty\colon A \to \mathcal{L}(E_\infty)$, $h_\infty(a) := h(a) \oplus h(a) \oplus \cdots$, has the property that the set C of c.p. maps $V_e\colon A \to B$ with $V_e(a) := \langle e, h_\infty(a)(e)\rangle$ is an operator convex cone.

Lemma 2.3. *Suppose that $\mathcal{C} \subset CP(A, B)$ is a point-norm closed operator convex cone. Then there is a Hilbert B-module E and a non-degenerate *-homomorphism $h \colon A \to \mathcal{L}(E)$ such that $\mathcal{C}$ is the point-norm closure of the maps $V_e \colon A \to B$ for $e \in E$, where $V_e(a) := \langle e, h(a)e \rangle$.*

If A is separable and $\mathcal{C}$ is countably generated, one can manage that, in addition, E is countably generated as Hilbert B-module.

Proof. Let $\mathcal{S}$ a subset of $\mathcal{C}$ that generates $\mathcal{C}$, and let E^T and $h^T \colon A \to \mathcal{L}\left(E^T\right)$ as in Lemma A.24 for $T \in \mathcal{S}$. Define F as the Hilbert B-module sum

$$F := \bigoplus_{T \in \mathcal{S}} E^T$$

with right A-module structure given by the non-degenerate *-homomorphism

$$g \colon a \in A \mapsto \bigoplus_{T \in \mathcal{S}} h^T(a).$$

The point-norm closure of the set of c.p. maps $V_e(a) := \langle e, g(a)e \rangle$ contains $\mathcal{S}$, and is contained in $\mathcal{C}$ by Lemma A.24 and Definition 2.1. Thus, by Remark 2.2, $E := F_\infty = F \oplus F \oplus \cdots$ and the non-degenerate *-homomorphism $h(a) := g_\infty(a) = g(a) \oplus g(a) \oplus \cdots$ are as desired.

If A is separable and $\mathcal{S}$ is countable, then E is countably generated as a Hilbert B-module, because then E^T is countably generated for $T \in \mathcal{S}$ by Lemma A.24. ∎

Remark 2.4. A C*-algebra B is stable if and only if there is a sequence of isometries $s_1, s_2, \ldots \in \mathcal{M}(B)$ such that $\sum s_n s_n^*$ converges strictly to 1. (Then automatically $s_j^* s_k = \delta_{j,k} 1$.)

The *infinite repeat* is (up to unitary equivalence) the unital endomorphism given by

$$\delta_\infty \colon d \in \mathcal{M}(B) \mapsto \sum s_i d s_i^* \in \mathcal{M}(B).$$

It is not hard to check that

(i) if $t_1, t_2, \ldots$ is a second sequence of isometries with $\sum t_i t_i^*$ converging strictly to 1, then δ_∞ and $\delta'_\infty \colon d \mapsto \sum t_i d t_i^*$ are unitarily equivalent by the unitary $U := \sum s_n t_n^* \in \mathcal{M}(B)$,

(ii) $\delta_\infty \circ \delta_\infty$ is unitarily equivalent to δ_∞.

(iii) Also note that $\delta_\infty(\mathcal{M}(B)) \cap B = \{0\}$, because $\delta_\infty(b) \in B$ implies $b = s_n^* \delta_\infty(b) s_n \to 0$ for $n \to \infty$.

Cf. [15, rem. 5.1.2, lem. 5.1.3] for details.

Remark 2.5. $\mathcal{H}_A$ and A are isomorphic as Hilbert A-modules if and only if A is stable.

Indeed: $\mathcal{L}(\mathcal{H}_A) \cong \mathcal{M}(\mathbb{K}(\mathcal{H}_A))$ always contains a sequence of isometries $s_1, s_2, \ldots$ such that $\sum_n s_n s_n^*$ strictly converges to 1 in $\mathcal{M}(\mathbb{K}(\mathcal{H}_A))$, because $\mathcal{H}_A \cong \mathcal{H}_A \oplus \mathcal{H}_A \oplus \cdots$. An Hilbert A-module isomorphism from $\mathcal{H}_A$ onto A defines an isomorphism from $\mathbb{K}(\mathcal{H}_A)$ onto $\mathbb{K}(A_A) \cong A$.

By Remark 2.4, A is stable if and only if there is a sequence of isometries $s_1, s_2, \ldots \in \mathcal{M}(A)$ with $\sum s_n s_n^*$ strictly convergent to 1.

The maps $a \in A \mapsto (s_1^* a, s_2^* a, \ldots)$ and $(a_1, a_2, \ldots) \in \mathcal{H}_A \mapsto \sum_n s_n a_n$ preserve the A-valued sesquilinear forms, are right A-module maps, and are inverse to each other.

Remark 2.6. For a separable and stable C*-algebra B every countably generated Hilbert B-module E is isomorphic to pB for a projection $p \in \mathcal{M}(B)$ with B-valued product $\langle a, b\rangle := a^* b$.

Thus: If A is separable, then every non-degenerate left A-module structure on E is given (up to isometric bi-module isomorphisms) by a non-degenerate *-homomorphism $h\colon A \to \mathcal{M}(pBp) \cong p\mathcal{M}(B)p$ and $E := pB$. (Use Remark 2.5 and Examples A.17(iv) and A.19.)

Remark 2.7. Suppose that B is stable and σ-unital and that D is a corner of B (i.e. there is a projection $p = p^* \in \mathcal{M}(B)$ with $pBp = D$). Then:
D is stable and full (i.e. the span of BDB is dense in B), if and only if, there is an isometry $v \in \mathcal{M}(B)$ with $vv^* = p$.

Indeed, by [3, Lemma 2.5] there are the Murray–von-Neumann equivalences $1 \otimes e_{1,1} \sim 1 \otimes 1 \sim p \otimes 1 \sim p \otimes e_{1,1}$ in $\mathcal{M}(B \otimes \mathbb{K})$. Note here that pbp is a strictly positive element of pBp if $b \in B_+$ is a strictly positive element of B.

Proposition 2.8. *Suppose that A and B are stable and separable* C**-algebras, and that $\mathcal{C}$ is a point-norm closed full operator convex cone of completely positive maps from A into B in the sense of Definition 2.1.*

*Then there is a non-degenerate *-homomorphism $h\colon A \to \mathcal{M}(B)$ with:*

(i) h is unitarily equivalent to $\delta_\infty \circ h$,

(ii) the c.p. maps $V_b\colon a \mapsto b^* h(a) b$ are in $\mathcal{C}$ for every $b \in B$, and

(iii) for every $V \in \mathcal{C}$ there is a sequence $b_n \in B$ such that $\|b_n\|^2 \le \|V\|$ and $\lim_n b_n^* h(a) b_n = V(a)$ for all $a \in A$.

Proof. $\mathcal{C}$ is separable in the point-norm topology, because A and B are separable. Thus $\mathcal{C}$ is countably generated. By Lemma 2.3, there is a countably generated Hilbert B-module E and a *-homomorphism $h_1\colon A \to \mathcal{L}(E)$ such that $\mathcal{C}$ is the point-norm closure of the set of maps V_e with $e \in E$, and $h_1(A)E$ is dense in E.

By Remark 2.6, there are a projection $p \in \mathcal{M}(B)$ and an isometric B-module isomorphism ι from E onto pB. Then $\iota(e)^*\iota(f) = \langle e, f\rangle$ and $h_2(a) := \iota \circ h_1(a) \circ \iota^{-1}$ defines a *-homomorphism from A into $\mathcal{L}(pB) \cong \mathcal{M}(pBp) \cong p\mathcal{M}(B)p$ with $h_2(A)pBp = pBp$. Thus, h_2 uniquely extends to a unital and strictly continuous *-homomorphism $\mathcal{M}(h_2)$ from $\mathcal{M}(A)$ onto $\mathcal{M}(pBp)$. Since A is stable, there is a sequence $t_1, t_2, \ldots$ of isometries in $\mathcal{M}(A)$ such that $\sum_n t_n t_n^*$ converges strictly to 1. The same happens with the isometries $s_n := \mathcal{M}(h_2)$ in $\mathcal{M}(pBp)$. It follows that pBp is a stable corner of B.

The fullness of $\mathcal{C}$ and the properties of E and h_1 imply that every $c \in B$ is in the closed linear span of the elements $b^*ph_2(a)pb$ with $a \in A$ and $b \in B$. It yields that pBp is also a full corner of B.

By Remark 2.7 there is an isometry $v \in \mathcal{M}(B)$ with $vv^* = p$.

Let $h_3(a) := v^*h_2(a)v$, then $h_3\colon A \to \mathcal{M}(B)$ is a non-degenerate *-homomorphism and satisfies (ii) and (iii) (with h_3 in place of h).

By Remark 2.5, the bi-module given by B_B and $\delta_\infty \circ h_3\colon A \to \mathcal{M}(B)$ is isomorphic to the bi-module given by $\mathcal{H}_B$ and $(h_3)_\infty\colon a \mapsto h_3(a) \oplus h_3(a) \oplus \cdots \in \mathcal{L}(\mathcal{H}_B)$. Thus $h := \delta_\infty \circ h_3\colon A \to \mathcal{M}(B)$ still satisfies (ii) and (iii) by Remark 2.2.

Since $\delta_\infty \circ \delta_\infty$ is unitarily equivalent to δ_∞ (cf. Remark 2.4), we get that $h := \delta_\infty \circ h_3$ satisfies (i)–(iii). $\square$

Definition 2.9. $\mathcal{M}(B)$ denotes the multiplier C*-algebra of a C*-algebra B. For any closed ideal J of B, $\mathcal{M}(B, J)$ means the set

$$\mathcal{M}(B, J) := \{t \in \mathcal{M}(B) :\ tB \subset J\} \subset \mathcal{M}(B)$$

of relative multipliers which multiply B into J.

Observe that $\mathcal{M}(B, J)$ is a strictly closed ideal of $\mathcal{M}(B)$ and that $J = \mathcal{M}(B, J)\cdot B = \mathcal{M}(B, J) \cap B$. Moreover $\mathcal{M}(B, J)$ is the kernel of the natural strictly continuous *-homomorphism $\mathcal{M}(\pi_J)$ from $\mathcal{M}(B)$ into $\mathcal{M}(B/J)$.

Remark 2.10. The *-homomorphism $h\colon A \to \mathcal{M}(B)$ of Proposition 2.8 defines a map $\Psi_h\colon \mathbb{I}(B) \to \mathbb{I}(A)$ by $\Psi_h(J) := h^{-1}(h(A) \cap \mathcal{M}(B, J))$. Then

$\Psi_h(B) = A$ and $\Psi_h(\{0\}) = \ker(h)$, and, by (i) and (ii),

$$\Psi_h(J)_+ = \{a \in A_+ : \ V(a) \in J \quad \text{for all } V \in \mathcal{C}\}.$$

It follows that $\Psi_h(\bigcap_\alpha J_\alpha) = \bigcap_\alpha \Psi_h(J_\alpha)$ for every family $\{J_\alpha\}$ of closed ideals of B. I.e. Ψ_h satisfies property (II) of Definition 1.1 if translated to $\mathbb{O}\,(\mathrm{Prim}\,(B)) \cong \mathbb{I}\,(B)$ and $\mathbb{O}\,(\mathrm{Prim}\,(A)) \cong \mathbb{I}\,(A)$. Further, h is faithful if and only if $\mathcal{C}$ is separating.

Definition 2.11. Let $h_j\colon A \to \mathcal{M}\,(B)$, $j = 1, 2$, *-homomorphisms from A into the multiplier algebra $\mathcal{M}\,(B)$ of B.

We call h_1 and h_2 *unitarily homotopic* if there is a norm-continuous map $t \mapsto U(t)$ from the non-negative real numbers $\mathbb{R}_+$ into the unitaries in $\mathcal{M}\,(B)$, such that, for $t \in \mathbb{R}_+$ and $a \in A$,

$$\begin{aligned} & U(t)^* h_1(a) U(t) - h_2(a) \in B \\ \text{and} \qquad & h_2(a) = \lim_{t\to\infty} U(t)^* h_1(a) U(t). \end{aligned}$$

The limit is taken in the norm of $\mathcal{M}\,(B)$.

Remark 2.12. $h\colon A \to \mathcal{M}\,(B)$ with (i)–(iii) of Proposition 2.8 is unique up to unitary homotopy (cf. Definition 2.11 and [15, cor. 5.1.6]).

2.2 Construction of A and $h\colon A \to \mathcal{M}\,(A)$ from Ψ

Lemma 2.13. *Suppose that D is a σ-unital hereditary* C**-subalgebras of a stable σ-unital* C**-algebra B such that D is full in B (i.e.* $\mathrm{span}\,(BDB)$ *is dense in B). Then there is an isomorphism φ from $D \otimes \mathbb{K}$ onto B such that $\varphi\,((D \cap I) \otimes \mathbb{K}) = I$ for all $I \in \mathbb{I}\,(B)$.*

If, in addition, D is stable, then there is an isomorphism ψ from D onto B with $\psi(D \cap I) = I$ for all $I \in \mathbb{I}\,(B)$.

(Special case of the Ψ-equivariant version of the stable isomorphism theorem of Brown [3], cf. [15, cor. 5.2.6]).

Proof. One can modify the proof in [3] as follows: We take a sequence of isometries $s_1, s_2, \ldots \in \mathcal{M}\,(B)$ with $\sum s_n s_n^*$ strictly convergent to 1, and strictly positive contractions $b \in B_+$ and $d \in D_+$.

Let $e := \sum_n 2^{-n} s_n d s_n^*$ and let $e_{j,k}$ denote matrix units of $\mathbb{K}$. It is easy to check that

$$\sum_{j,k} a_{j,k} \otimes e_{j,k} \mapsto \sum_{j,k} s_j a_{j,k} s_k^*$$

extends to a *-monomorphism τ from $D \otimes \mathbb{K}$ onto the hereditary C*-subalgebra $D_0 := \overline{eBe}$ of B and satisfies $\tau\left((D \cap I) \otimes \mathbb{K}\right) = D_0 \cap I$ for closed ideals I of B. In particular, D_0 is full in B if and only if D is full in B. Thus it suffices to find an isomorphism ψ_0 from the stable D_0 onto B with $\psi_0\left(D_0 \cap I\right) = I$ for $I \in \mathbb{I}(B)$, because then $\varphi := \psi_0 \circ \tau$ is as desired.

Suppose now that D is a stable full and σ-unital hereditary C*-subalgebra of B (i.e. $DBD = D$, $D \cong D \otimes \mathbb{K}$, span (BDB) is dense in B and D_+ contains a strictly positive element d). Let $d_1 := s_1 b s_1^*$, $d_2 := s_2 d s_2^*$, $d_3 := d_1 + d_2$, and let D_k be the hereditary C*-subalgebra of B generated by d_k ($k = 1, 2, 3$). Then D_k is a stable σ-unital full hereditary C*-subalgebra of B for $k = 1, 2, 3$, $D_1 = s_1 B s_1 *$ and $D_2 = s_2 D s_2^*$. Moreover, D_1 and D_2 are orthogonal corners of D_3 such that $D_1 + D_2$ contains the strictly positive element d_3 of D_3. Thus D_1 and D_2 are σ-unital, stable and full corners of D_3. By Remark 2.7, there are isometries $t_1, t_2 \in \mathcal{M}(D_3)$ with $D_j = t_j t_j^* D_3 t_j t_j^* = t_j D_3 t_j^*$ for $j = 1, 2$. Let $v := t_1 t_2^* \in \mathcal{M}(D_3)$ and $\psi(a) := s_1^* v (s_2 a s_2^*) v^* s_1$ for $a \in D$. ψ is an isomorphism from D onto B and satisfies $\psi(D \cap I) = I$, because $s_2(D \cap I)s_2^* = D_2 \cap I$, $s_1^*(I \cap D_1)s_1 = I$, $D_k \cap I = D_k \cap (D_3 \cap I)$. $\square$

Definition 2.14. Let $\Psi\colon \mathbb{I}(B) \to \mathbb{I}(A)$ an order preserving map. A completely positive map $V\colon A \to B$ is Ψ-*equivariant* if $V(\Psi(J)) \subset J$ for every $J \in \mathbb{I}(B)$. V is Ψ-*residually nuclear* if V is Ψ-equivariant and the induced maps $[V]_J\colon A/\Psi(J) \to B/J$ are nuclear for every $J \in \mathbb{I}(B)$.

Clearly, every Ψ-equivariant c.p. map $V\colon A \to B$ is Ψ-residually nuclear if A or B is nuclear. But for non-nuclear A and B the nuclearity of c.p. maps with $V(\Psi(J)) \subset J$ does not imply the nuclearity of $[V]_J$ in general (cf. [15, sec. 5.3]).

Proposition 2.15. *Suppose that A and B are separable stable* C*-*algebras such that $B \otimes \mathcal{O}_2$ contains an Abelian regular* C*-*subalgebra C (cf. Definition 1.2). Then every map*

$$\Psi\colon \mathbb{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \to \mathbb{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$$

*with properties (I) and (II) of Definition 1.1 can be realized by a *-monomorphism $h\colon A \hookrightarrow \mathcal{M}(B)$ with the following properties:*

(i) h *is non-degenerate, i.e.* $h(A)B$ *is dense in* B.

(ii) h *is unitarily equivalent to its infinite repeat* $\delta_\infty \circ h$.

(iii) $\Psi(J) = h^{-1}(h(A) \cap \mathcal{M}(B, J))$ *for every* $J \in \mathbb{I}(B)$.

(iv) For every $b \in B$ *the completely positive map* $T_b \colon a \in A \mapsto b^* h(a) b \in B$ *is* Ψ*-residually nuclear.*

Proof. First we consider the case where $B = C_0(Y, \mathbb{K})$ for a locally compact Polish space Y. The set $\mathcal{C}$ of Ψ-equivariant c.p. maps T from A to B is closed in point-norm, and is operator convex in the sense of Definition 2.1. Every $T \in \mathcal{C}$ is Ψ-residually nuclear, because $C_0(Y, \mathbb{K})$ is nuclear.

If we use the correspondence between $\mathbb{I}(B)$ and $\mathbb{O}(Y)$, then Lemma A.15 tells us that for every $J \in \mathbb{I}(B)$ and every $a \notin \Psi(J)$ there is $T \in \mathcal{C}$ with $T(a) \notin J$. Thus,

$$\Psi(J) = \{a \in A : \quad V(a) \in J \quad \text{for all } V \in \mathcal{C}\}.$$

This implies $\Psi(J_0) = A$ for the closed linear span J_0 of $\{T(a) : \quad T \in \mathcal{C},\, a \in A\}$. Since $\Psi^{-1}(A) = \{B\}$ (by property (II) of Ψ), it follows $J_0 = B$, i.e. $\mathcal{C}$ is full.

By Proposition 2.8 there is a non-degenerate *-homomorphism $h \colon A \to \mathcal{M}(B)$ with properties (i)–(iv). Since our $\mathcal{C}$ is separating for $A_+ \setminus \{0\}$, it follows that h is faithful by Remark 2.10.

The case of general B and Abelian regular $C \subset B$ reduces to the above special case as follows:

Let B a separable stable C*-algebra and $C \subset B$ an Abelian regular C*-subalgebra. In particular, the hereditary C*-subalgebra $D := \overline{CBC}$ of B is full. Thus, by Lemma 2.13 there is an isomorphism φ from $D \otimes \mathbb{K}$ onto B with $\varphi((D \cap J) \otimes \mathbb{K}) = J$ for $J \in \mathbb{I}(B)$.

It holds $C \cong C_0(Y)$ for the locally compact space $Y := \operatorname{Prim}(C)$ (by Gelfand transformation), and the restriction of φ to $C \otimes \mathbb{K}$ defines a non-degenerate *-monomorphism η from $C_0(Y, \mathbb{K}) \cong C \otimes \mathbb{K} \subset D \otimes \mathbb{K}$ into B. Let $B_1 := \eta(C_0(Y, \mathbb{K})) = \varphi(C \otimes \mathbb{K})$. Then

$$B_1 \cap J = \varphi((C \cap J) \otimes \mathbb{K}) = \eta(C_0(U, \mathbb{K}))$$

for $J \in \mathbb{I}(B)$ and the support U of $C \cap J$ in $Y = \operatorname{Prim}(C)$.

Let $\Psi_1 \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathbb{O}(Y)$ the lattice monomorphism (with properties (I)–(IV) of Definition 1.1) that is induced by $I \mapsto \eta^{-1}(I \cap \varphi(B_1))$ (cf. Lemma A.14). We use the right-inverse Φ_1 of Ψ_1 as considered in Definition A.2 and Lemma A.3, and define a map $\Phi_2 \colon \mathbb{O}(Y) \to \mathbb{I}(B)$ by $\Phi_1(U) := \mathsf{k}(\mathrm{Prim}(B) \setminus \Phi_1(U))$. Then Φ_1 satisfies property (I) and (II) of Definition 1.1 by Lemma A.3. Further $\Phi_2(U) = J$ for the support U of $I := \eta^{-1}(B_1 \cap J)$, because $C \cap J$ and $I \subset C_0(Y, \mathbb{K})$ have the same support in $Y = \mathrm{Prim}(C)$ and $\Psi_1(V) = U$ for the support $V \in \mathbb{O}(\mathrm{Prim}(B))$ of J.

Let $\Psi_3(U) := \Psi(\Phi_2(U))$ for $U \in \mathbb{O}(Y)$. Ψ_3 satisfies properties (I) and (II), because Ψ and Ψ_2 satisfy (I) and (II), and $\Psi_3(U) = \Psi(J)$ for $J \in \mathbb{I}(B)$ and for the support U of $\eta^{-1}(B_1 \cap J)$. For $I \in \mathbb{I}(B_1)$ let $\Psi_4(I) := \Psi_3(U)$ for the support $U \in \mathbb{O}(Y)$ of $\eta^{-1}(I) \in C_0(Y, K)$. Then $\Psi_4(B_1 \cap J) = \Psi(J)$ for $J \in \mathbb{I}(B)$.

By the above considered special case, there is a *-monomorphism $h \colon A \to \mathcal{M}(B_1)$ with (i)–(iv) (for (B_1, Ψ_4) in place of (B, Ψ)).

The nuclear C*-subalgebra $B_1 \subset B$ satisfies $\mathrm{span}(B_1 B)$ dense in B. Thus, $\mathcal{M}(B_1)$ is (in a natural way) a strictly closed C*-subalgebra of $\mathcal{M}(B)$.

Then $h \colon A \to \mathcal{M}(B)$ is a monomorphism and satisfies (i)–(iii):

$h(A)B = h(A)B_1B = B_1B = B$, h is unitarily equivalent to $\delta_\infty \circ h$ by Remark 2.4(ii), and $h(A) \cap \mathcal{M}(B, J) = h(\Psi(J))$ because

$$\mathcal{M}(B, J) \cap \mathcal{M}(B_1) = \mathcal{M}(B_1, B_1 \cap J)$$

and $\Psi_4(B_1 \cap J) = \Psi(J)$.

$V_b \colon a \mapsto b^* h(a) b$ is a Ψ-equivariant c.p. map from A into B by (iii). $[V_b]_J \colon A/\Psi(J) \to B/J$ factorizes over the nuclear C*-algebra $B_1/(B_1 \cap J)$, because

$$[V_b]_J(a + \Psi(J)) = \pi_J(d)^* [V_c]_I(a + \Psi(J)) \pi(d)$$

for $I := B_1 \cap J$, $c \in B_1$ and $d \in B$ with $cd = b$. Thus V_b is also residually nuclear for every $b \in B$, i.e. h satisfies also (iv) for Ψ and B. □

Remark 2.16. h has the property that a completely positive map V from A to B is Ψ-residually nuclear (cf. Definition 2.14) if and only if V can be approximated in point-norm by completely positive maps W_b for a suitable $b \in B$, cf. [15, chp. 3]. It follows that the map h of Proposition 2.15 is determined by (i)–(iv) up to unitary homotopy, cf. Remark 2.12.

Remark 2.17. By [15, chp. 1, cor. L], every isomorphism α from the primitive ideal space $\mathrm{Prim}(A)$ of a separable nuclear C*-algebra A onto the primitive ideal space $\operatorname{Prim}(B)$ of a separable nuclear C*-algebra B is induced by an *-isomorphism φ from $A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ onto $B \otimes \mathcal{O}_2 \otimes \mathbb{K}$.

Corollary 2.18. *Let P be a locally compact Polish space and $A := C_0(P, \mathbb{K})$. Suppose that Ω is a sublattice of $\mathbb{O}(P) \cong \mathbb{I}(A)$ which is closed under l.u.b. and g.l.b. and contains $\emptyset \sim \{0\}$ and $P \sim A$. Then there exists a *-monomorphism $h \colon A \to \mathcal{M}(A)$ with the following properties:*

(i) h is non-degenerate and faithful,

(ii) h is unitarily equivalent to its infinite repeat $\delta_\infty \circ h$,

(iii) if $J \in \mathbb{I}(A)$ satisfies $h(J)A \subset J$ then $h(A) \cap \mathcal{M}(A, J) = h(J)$,

(iv) the support of $J \in \mathbb{I}(A)$ is in Ω if and only if $h(J)A \subset J$

Proof. By Remark A.4 there is a map $\Psi \colon \mathbb{O}(P) \to \mathbb{O}(P)$ that satisfies properties (I) and (II) of Definition 1.1, $\Psi \circ \Psi = \Psi$, $\Psi(V) \subset V$ for $V \in \mathbb{O}(P)$ and $\Psi(V) = V$ if and only if $V \in \Omega$.

Recall that $P = \operatorname{Prim}(A)$, $\mathsf{k}(P \setminus U) = C_0(U, \mathbb{K}) \in \mathbb{I}(A)$ for $U \in \mathbb{O}(P)$, and $J = C_0(U_J, \mathbb{K})$ for the support $U_J \in \mathbb{O}(P)$ and $J \in \mathbb{I}(A)$. By Proposition 2.15 there exists a non-degenerate *-monomorphism $h \colon A \hookrightarrow \mathcal{M}(A)$ such that

$$C_0(\Psi(V), \mathbb{K}) = h^{-1}(h(A) \cap \mathcal{M}(A, C_0(V, \mathbb{K})))$$

for open subsets V of P and $\delta_\infty \circ h$ unitarily equivalent to h. Thus, for $J = C_0(U_J, \mathbb{K})$ holds $h(J)A \subset J$, i.e. $J \subset h^{-1}(h(A) \cap \mathcal{M}(A, J))$, if and only if $U_J \subset \Psi(U_J)$. The latter implies $U_J = \Psi(U_J)$, i.e. $h(J) = h(A) \cap \mathcal{M}(A, J)$. Hence, h satisfies (i)-(iv). $\square$

Remark 2.19. Suppose that A is C*-algebra and $h \colon A \to \mathcal{M}(A)$ a non-degenerate *-monomorphism (i.e. $h^{-1}(0) = \{0\}$ and $h(A)A = A$). Consider the set Ω of supports of closed ideals J with the property $h(J)A \subset J$. Then Ω is a sub-lattice of $\mathbb{O}(\operatorname{Prim}(A))$ that contains $\emptyset$ and $\operatorname{Prim}(A)$ and is closed under l.u.b. (=unions) and g.l.b. (=interiors of intersections).

Indeed, $h(J_\alpha) A \subset J_\alpha$ implies $h(\bigcap_\alpha J_\alpha) A \subset \bigcap_\alpha J_\alpha$, i.e. Ω is closed under kernels of intersections (g.l.b.). Similarly,

$$h\left(\overline{\sum_\alpha J_\alpha}\right) A \subset \overline{\sum_\alpha J_\alpha},$$

i.e. that Ω is also closed under unions (l.u.b.).

Remark 2.20. The assumption of Corollary 2.18 implies, that for closed ideals J of A with $h(J)A \subset J$ holds

$$h(J) = h(A) \cap \mathcal{M}(A, J) = h(A) \cap (\mathcal{M}(A, J) + A)\,, \tag{2.1}$$

i.e. if $a, b \in A$ and $(h(a) + b)\,A \subset J$ then $a, b \in J$.

Indeed, if $U \in \mathcal{M}(A)$ is a unitary with $U^*h(a)U = \delta_\infty(h(a))$ and $c := U^*bU \in A$, then $(\delta_\infty(h(a)) + c)\,A \subset J$. For elements $e \in A$ we get $h(a)e + s_j^* c s_j e \in J$ and $\lim_{j\to\infty} \|cs_j\| = 0$ because $\sum s_j s_j^*$ converges strictly to 1 in $\mathcal{M}(A)$. It follows that $h(a) \in \mathcal{M}(A, J)$, thus $a \in J$ by (iii).

In a similar way it holds for $d \in \mathcal{M}(A)$, that $\delta_\infty(d) \in (\mathcal{M}(A, J) + A)$, if and only if, $d \in \mathcal{M}(A, J)$, if and only if, $\delta_\infty(d) \in \mathcal{M}(A, J)$.

Remark 2.21. Let A be a C*-algebra and h a non-degenerate *-monomorphism from A into $\mathcal{M}(A)$ such that $h(A) \cap A = \{0\}$. Then h uniquely extends to a faithful unital strictly continuous endomorphism of the multiplier algebra $\mathcal{M}(A)$, which we denote also by h.

If A is a type I C*-algebra then clearly the closure E of

$$A + h(A) + h^2(A) + h^3(A) + \dots$$

is a type I C*-algebra, that has a decomposition series with intermediate factors isomorphic to A. h defines a non-degenerate endomorphism of E. If we take the inductive limit $F = \mathrm{indlim}\,(h\colon E \to E)$ then there is a natural isomorphism σ of F such that (under canonical identification) $\sigma^{-1}(a) = h(a)$ for $a \in A \subset E \subset F$. Let $D = h(A) \subset E \subset F$. D satisfies $D\sigma(D) \subset \sigma(D)$, that $\sigma(D)$ is an essential ideal of $D + \sigma(D)$, and $D \cap \sigma(D) = \{0\}$. Then E is naturally isomorphic to $D_{-\infty,1}$, cf. Remark 4.12.

We give an explicit description of the corresponding embeddings and the isomorphism σ by natural embedding of F into sequence spaces modulo zero sequences:

Let $B := \ell_\infty(\mathcal{M}(A))\,/c_0(\mathcal{M}(A))$, $\sigma \in \mathrm{Aut}\,(B)$ induced by the forward shift on $\ell_\infty(\mathcal{M}(A))$,

$$\sigma\left((m_1, m_2, \dots) + c_0(\mathcal{M}(A))\right) = (0, m_1, m_2, \dots) + c_0(\mathcal{M}(A))\,.$$

The inductive limit $\mathrm{indlim}\,(h\colon \mathcal{M}(A) \to \mathcal{M}(A)) \supset F \supset E$ embeds canonically into B:

$$\begin{tikzcd}
\mathcal{M}(A) \arrow[r, "h"] & \mathcal{M}(A) \arrow[r, "h"] & \mathcal{M}(A) \arrow[r, "h"] & \cdots \arrow[r, dotted] & B \\
A \arrow[r] \arrow[ur, "h"] & A + h(A) \arrow[r] \arrow[ur, "h"] & \cdots & \arrow[r, dotted] & E \arrow[u, hook]
\end{tikzcd}$$

Then E is embedded in B and $\sigma^{-1}(c)$ equals $h(c)$ for $c \in E$, and F is the smallest σ-invariant C*-subalgebra of B containing $D := h(A) \subset E$. We shall see below that the crossed product $F \rtimes_\sigma \mathbb{Z}$ is isomorphic to the Cuntz–Pimsner algebra of the Hilbert bi-module $\mathcal{H}(A, h)$ given by $h \colon A \to \mathcal{M}(A)$ (cf. (i) of Theorem 1.4).

Remark 2.22. If $h \colon A \to \mathcal{M}(A)$ is as in Remark 2.21 and every closed ideal J of A with $h(J)A \subset J$ satisfies Equation (2.1) then the lattice of closed ideals of $E \rtimes_\sigma \mathbb{Z}$ is naturally isomorphic to the sub-lattice Ω of closed ideals J of A with $h(J)A \subset J$ (cf. (ii) of Theorem 1.4).

3 Cuntz–Pimsner algebras

3.1 Fock bi-module and Toeplitz algebra

In [24] Pimsner has defined two algebras, namely the Toeplitz algebra $\mathcal{T}(E)$ and the Cuntz–Pimsner algebra $\mathcal{O}(E)$ of a Hilbert A-bi-module E. Those algebras are defined as a C*-subalgebra and a C*-sub-quotient of the adjoint-able bounded operators over the (generalized) *Fock space* $\mathcal{F}(E)$ which is the Hilbert A-module

$$\mathcal{F}(E) := \bigoplus_{n=1}^{\infty} E^{(\otimes_A)n},$$

where

$$E^{(\otimes_A)0} := A_A\,, \ \ E^{(\otimes_A)1} := E\,, \ \ \text{and} \ \ E^{(\otimes_A)n} := \underbrace{E \otimes_A E \otimes_A \cdots \otimes_A E}_{n \text{ factors}}.$$

For the precise Definition of $E^{(\otimes_A)n}$ see Remark A.20, where also the maps $\eta_n \colon A \to \mathcal{L}\left(E^{(\otimes_A)n}\right)$ and left A-module structures $a \cdot e := \eta_n(h(a))e$ on $E^{(\otimes_A)n}$ are defined. (Here $E^{(\otimes_A)0} = A_A$ is as in Example A.17(i) and the left multiplication is defined by $a \cdot e := ae$.)

The For $e \in E$, let T_e denote the operator

$$T_e\,(a, f_1, f_2, \ldots) := (0, ea, e \otimes_A f_1, e \otimes_A f_2, \ldots)\ , \tag{3.1}$$

where $a \in A$ and $f_n \in E^{(\otimes_A)n}$ with $\sum_n \langle f_n, f_n\rangle$ convergent in A. Then, by Remark A.20(iii), $\|T_e\| \leq \|e\|$ and T_e is adjoint-able with adjoint $(T_e)^*$ given by

$$(T_e)^*\,(a, e_1, e_2 \otimes_A f_1, e_3 \otimes_A f_2, \ldots) = (\langle e, e_1\rangle\,, \langle e, e_2\rangle \cdot f_1, \langle e, e_3\rangle \cdot f_2, \ldots)\ ,$$

where $e_n \in E$, $f_n \in E^{(\otimes_A)n}$ with $\sum_n \langle e_{n+1} \otimes_A f_n, e_{n+1} \otimes_A f_n \rangle \in A$.

Definition 3.1. The (generalized) *Toeplitz algebra* is the C*-subalgebra $\mathcal{T}(E)$ of $\mathcal{L}(\mathcal{F}(E))$ generated by operators T_e for all $e \in E$.

Let $p_k(a, f_1, \ldots, f_k, f_{k+1} \ldots) := (a, f_1, \ldots, f_k, 0, 0, \ldots)$ for $k \in \mathbb{N}$. p_k is a self-adjoint projection in $\mathcal{L}(\mathcal{F}(E))$ and $p_k \leq p_{k+1}$.

$\mathcal{J}(E)$ denotes the hereditary C*-subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\{p_1, p_2, \ldots\}$, i.e. the closure of $\bigcup_k p_k \mathcal{L}(\mathcal{F}(E)) p_k$.

Remark 3.2. $\mathcal{J}(E)$ is essential in $\mathcal{L}(\mathcal{F}(E))$, because the (left) $\mathcal{J}(E)$-module E is (obviously) non-degenerate. Thus, the (two-sided) normalizer algebra $\mathcal{N}(\mathcal{J}(E))$ of $\mathcal{J}(E)$ in $\mathcal{L}(\mathcal{F}(E))$ is in a natural manner a unital C*-subalgebra of the multiplier algebra $\mathcal{M}(\mathcal{J}(E))$ of $\mathcal{J}(E)$.

Since $p_{k+1} T_e p_k = T_e p_k$ and $p_k T_e = p_k T_e p_{k-1}$, T_e is in the normalizer algebra of $\mathcal{J}(E)$. Thus the Toeplitz algebra $\mathcal{T}(E) \subset \mathcal{L}(\mathcal{F}(E))$ is naturally contained in $\mathcal{M}(\mathcal{J}(E))$.

Definition 3.3. The *Cuntz–Pimsner algebra* $\mathcal{O}(E)$ is defined as the image of the Toeplitz algebra $\mathcal{T}(E)$ in the corona algebra $\mathcal{M}(\mathcal{J}(E)) / \mathcal{J}(E)$ of $\mathcal{J}(E)$.

Remark 3.4. The natural *-monomorphism from $\mathcal{N}(\mathcal{J}(E))$ to the multiplier algebra $\mathcal{M}(\mathcal{J}(E))$ is an *isomorphism* from $\mathcal{N}(\mathcal{J}(E))$ *onto* $\mathcal{M}(\mathcal{J}(E))$, because, if a bounded net $\{S_\alpha\} \subset \mathcal{J}(E)$ converges strictly to an element S of $\mathcal{M}(\mathcal{J}(E))$, then the net $\{S_\alpha\}$ converges in $\mathcal{L}(\mathcal{F}(E))$ strongly to a multiplier of $\mathcal{J}(E)$ (cf. the remark following Definition A.18).

3.2 The Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H}(A, h))$

We give another description of the Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H}(A, h))$ of a bi-module $\mathcal{H}(A, h)$ corresponding to a *non-degenerate* *-homomorphism $h\colon A \to \mathcal{M}(A)$ (cf. Definition A.22, compare also Examples A.21, A.19 and A.17).

h defines a full Hilbert A-bi-module $\mathcal{H}(A, h)$ with right Hilbert A-module $E := A_A$ considered in Examples A.17(i), and left A-module structure given by $h\colon A \to \mathcal{M}(A) = \mathcal{L}(E)$ (cf. A.19). The unique extension of h to a strictly continuous unital *-homomorphism from $\mathcal{M}(A)$ into $\mathcal{M}(A)$ will also be denoted by h (to keep notation simple). Then the restrictions of h^n to A are non-degenerate *-homomorphisms $h^n\colon A \to \mathcal{M}(A)$.

Remark 3.5. By Remark A.23, there are isomorphisms I_n from the the n-fold tensor products $E^{(\otimes A)n}$ of the Hilbert A-bi-module E given by the non-degenerate *-homomorphism $h\colon A \to \mathcal{M}(A)$ onto the Hilbert A-bi-module given by $h^n\colon A \to \mathcal{M}(A)$. Under this identifications $e \otimes_A f$ becomes $h^n(e)f$ for $e \in A \cong E$ and $f \in A \cong E^{(\otimes A)n}$. (Recall here that $E^{(\otimes A)0}$ is A_A with left multiplication given by $h^0\colon A \to \mathcal{M}(A)$ where $h^0(a)b := ab$.)

Hence, $\mathcal{F}(E)$ is isomorphic to $\mathcal{H}_A$ by an isomorphism such that T_e becomes $T_e(a_0, a_1, a_2, \ldots) := (0, ea_0, h(e)a_1, h^2(e)a_2, \ldots)$ for $(a_0, a_1, \ldots)$ in $\mathcal{H}_A$.

Since $\mathcal{H}_A$ is nothing else than $(A \otimes \mathbb{K})(1 \otimes e_{0,0})$ for the minimal projection $e_{0,0} \in \mathbb{K} = \mathbb{K}(\ell_2\{0, 1, 2, \ldots\})$, there is a natural isomorphism from $\mathcal{M}(A \otimes \mathbb{K})$ onto $\mathcal{L}(\mathcal{H}_A) \cong \mathcal{L}(\mathcal{F}(E))$ such that the hereditary C*-subalgebra $\mathcal{M}(A) \otimes \mathbb{K} \subset \mathcal{M}(A \otimes \mathbb{K})$ maps onto $\mathcal{J}(E)$ (cf. the remark following Definition A.18). Clearly $\ell_\infty(\mathcal{M}(A)) \subset \mathcal{N}(\mathcal{M}(A) \otimes \mathbb{K}) \cong \mathcal{M}(\mathcal{M}(A) \otimes \mathbb{K})$. Let, for $a \in \mathcal{M}(A)$,

$$h^\infty(a) := \big(a, h(a), h^2(a), h^3(a), \ldots\big) \in \ell_\infty(\mathcal{M}(A)) \subset \mathcal{L}(\mathcal{H}_A)\ ,$$

and let $\mathcal{T} \in \mathcal{L}(\mathcal{H}_A)$ denote the forward shift $\mathcal{T}(a_0, a_1, \ldots) := (0, a_0, a_1, \ldots)$, i.e. $\mathcal{T} = 1 \otimes \mathcal{T}_0$ where $\mathcal{T}_0$ is a Toeplitz operator (forward shift) on $\ell_2(\mathbb{N})$. Then $\mathcal{T}$ is in $\mathcal{N}(\mathcal{M}(A) \otimes \mathbb{K}) \cong \mathcal{M}(\mathcal{M}(A) \otimes \mathbb{K})$, and T_e decomposes as $T_e = \mathcal{T}h^\infty(e)$ for $e \in A$. Note that $U := \mathcal{T} + (\mathcal{M}(A) \otimes \mathbb{K})$ is a unitary in the stable corona $Q^s(\mathcal{M}(A)) := \mathcal{M}(\mathcal{M}(A) \otimes \mathbb{K}) / (\mathcal{M}(A) \otimes \mathbb{K})$ of $\mathcal{M}(A) \otimes \mathbb{K}$, that

$$h_{1,\infty}\colon a \in \mathcal{M}(A) \mapsto h^\infty(a) + (\mathcal{M}(A) \otimes \mathbb{K}) \in Q^s(\mathcal{M}(A))$$

is a *-homomorphism from $\mathcal{M}(A)$ into the corona $Q^s(\mathcal{M}(A))$ of $\mathcal{M}(A) \otimes \mathbb{K}$.

It follows that *the Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H}(A, h))$ is the* C*-*subalgebra* $\mathrm{C}^*(Uh_{1,\infty}(A))$ *of $Q^s(\mathcal{M}(A))$ which is generated by the elements $Uh_{1,\infty}(e)$ for $e \in A$.*

Recall that the Toeplitz operator $\mathcal{T}$ is related to the forward shift on $\ell_\infty(\mathcal{M}(A))$ as follows: Let $B := \ell_\infty(\mathcal{M}(A)) / c_0(\mathcal{M}(A)) \subset Q^s(\mathcal{M}(A))$ and σ denote the automorphism of B induced by the forward shift $(a_0, a_1, \ldots) \to (0, a_0, a_1, \ldots)$ on $\ell_\infty(\mathcal{M}(A))$. Then $UbU^* = \sigma(b)$ for $b \in B$ by Lemma A.27.

On the other hand, $h_{1,\infty}(h(a)) = \sigma^{-1}(h_{1,\infty}(a))$ for $a \in \mathcal{M}(A)$. Thus, $h_{1,\infty}(h(a)b) = \sigma^{-1}(h_{1,\infty}(a))\, h_{1,\infty}(b)\,,$ and, hence,

$$h_{1,\infty}(h(a)b) = U^{-1}h_{1,\infty}(a)Uh_{1,\infty}(b) \quad \textit{for}\ \ a, b \in \mathcal{M}(A)\,.$$

Proposition 3.6. *Suppose that $h\colon A \to \mathcal{M}(A)$ is a non-degenerate *-homomorphism. Let*

$$B := \ell_\infty(\mathcal{M}(A))/c_0(\mathcal{M}(A)) \subset Q^s(\mathcal{M}(A))$$

and $\sigma \in \mathrm{Aut}(B)$ induced by the forward shift on on $\ell_\infty(\mathcal{M}(A))$.

*There are a *-homomorphism $\varphi\colon A \to B$ and a unitary $U \in Q^s(\mathcal{M}(A))$ such that*

(i) $\sigma(b) = UbU^*$ for $b \in B$,

(ii) $\varphi(h(a)b) = U^*\varphi(a)U\varphi(b)$ for all $a, b \in A$, and

(iii) $U\varphi(A)$ generates the Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H}(A,h))$ as C^*-subalgebra of $Q^s(\mathcal{M}(A))$, in particular $\varphi(A) \subset \mathcal{O}(\mathcal{H}(A,h))$.

Let $D := \varphi(A) \subset B$ and let $[D]_\sigma$ denote the smallest σ-invariant C^*-subalgebra of B containing D. Then the C^*-subalgebra E of $Q^s(\mathcal{M}(A))$ that is generated by $\bigcup_{k\in\mathbb{N}} U^{-k}D$ is naturally isomorphic to $[D]_\sigma \rtimes_\sigma \mathbb{Z}$, and $\mathcal{O}(\mathcal{H}(A,h))$ is the full hereditary C^*-subalgebra of E generated by D.

$[D]_\sigma \rtimes_\sigma \mathbb{Z}$ and $\mathcal{O}(\mathcal{H}(A,h))$ are isomorphic if A is stable and contains a strictly positive element.

φ is a monomorphism if h is moreover faithful.

Proof. Let $U \in Q^s(\mathcal{M}(A))$, $\sigma \in \mathrm{Aut}(B)$, $\varphi := h_{1,\infty}$ and $D := \varphi(A)$, where $h_{1,\infty}$ and σ are as in Remark 3.5.

Then, by Remark 3.5, $D \subset B \subset Q^s(\mathcal{M}(A))$, U is a unitary in $Q^s(\mathcal{M}(A))$ with $UbU^{-1} = \sigma(b)$ for $b \in B$, and $\varphi^{-1}(D)D = D$, because $h_{1,\infty}(h(a)b) = \sigma^{-1}(h_{1,\infty}(a))\, h_{1,\infty}(b)$ and $h(A)A = A$.

$\sigma^{-k}(D)D = D$, because $\sigma^{-j}(D)\sigma^{1-j}(D) = \sigma^{1-j}(D)$ for $1 \le j \le k$. It means $U^{-k}DU^kD = D$, $DU^kD = U^kD$ and $DU^{-k}D = DU^{-k}$ for $k \in \mathbb{N}$. It shows for $i, j, m, n \in \mathbb{Z}$:

$$U^iDU^jU^mDU^n = \begin{cases} U^{i+(j+m)}DU^n & \text{if } j+m \ge 0, \\ U^iDU^{n+(j+m)} & \text{if } j+m < 0. \end{cases} \tag{3.2}$$

Thus, the sum $C := \sum_{i,j\in\mathbb{Z}} U^iDU^j$ is a *-subalgebra of $Q^s(\mathcal{M}(A))$, and U is in the two-sided normalizer $\mathcal{N}(C)$ of C. It follows $U \in \mathcal{N}(\overline{C})$.

C is *-algebraically generated by $\bigcup_{k\in\mathbb{N}} U^{-k}D$, because the C^*-algebra generated by $U^{-1}D, U^{-2}D, \ldots$ contains also $D = (U^*D)^*(U^*D)$, $DU^{-k}D =$

DU^{-k} and $U^kD = \left(DU^{-k}\right)^*$ for $k \in \mathbb{N}$, and finally, $(U^iD)\,(U^{-j}D)^* = U^iDU^j$ for $i, j \in \mathbb{Z}$.

$\sum_{j\in\mathbb{Z}} \sigma^j(D)$ is a *-subalgebra of $B \cap C$, because $U^kDU^{-k} = \sigma^k(D)$ and $\sigma^j(D)\sigma^k(D) = U^jDU^{k-j}DU^k = \sigma^{\max(j,k)}(D)$ by (3.2). Let $D_{-\infty,\infty}$ denote its closure. Then $D_{-\infty,\infty}$ is the smallest σ-invariant C*-subalgebra of B that contains D, i.e. $D_{\infty,\infty} = [D]_\sigma$. $\overline{C}$ is the C*-subalgebra of $Q^s(\mathcal{M}(A))$ generated by $UD_{-\infty,\infty} = D_{-\infty,\infty}U$, because C is also generated by $\bigcup_{k\in\mathbb{N}} \sigma^k(D)U$ for $k \in \mathbb{N}$ (note $U^{-k}D = \sigma^k(D)U \cdots \sigma^2(D)U\sigma(D)U$).

Since $C^*(B, U) \subset Q^s(\mathcal{M}(A))$ is naturally isomorphic to $B \rtimes_\sigma Z$ (by Lemma A.27), $\overline{C}$ is naturally isomorphic to $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ (by Remark A.25(iii)).

$G := \mathcal{O}(\mathcal{H}(A, h))$ is the C*-subalgebra of $Q^s(\mathcal{M}(A))$ that is generated by UD, cf. Remark 3.5. It follows $D = (UD)^*UD \subset G$, and $U^nD \subset G$ by induction, because $U^{n+1}D = U^nDUD$. Hence, $U^mDU^{-n} \subset G$ for $m, n \geq 0$. The sum $C_1 := \sum_{m,n\geq 0} U^mDU^{-n}$ is identical to the *-subalgebra of C given by DCD), because $DU^iDU^jD = U^mDU^{-n}$ with $m, n \geq 0$ for $i, j \in \mathbb{Z}$ by Equation (3.2). Thus, $\overline{C_1}$ is the hereditary C*-subalgebra of $\overline{C}$ generated by D, and $G = \overline{C_1}$.

The *-ideal of C generated by $D \subset C_1$ is is obviously identical with C. Thus G is the full hereditary C*-subalgebra of $\overline{C}$ generated by D.

$\|h_{1,\infty}(a)\| = \lim_{n\to\infty} \|h^n(a)\|$ for $a \in \mathcal{M}(A)$. $h^n\colon \mathcal{M}(A) \to \mathcal{M}(A)$ is faithful for every $n \in \mathbb{N}$ if $h\colon A \to \mathcal{M}(A)$ is faithful. Thus, $\|\varphi(a)\| = \|h^n(a)\| = \|a\|$ if h is faithful (in addition).

$E = \overline{C}$ and $\mathcal{O}(\mathcal{H}(A, h)) = DED$ contain strictly positive elements if A contains a strictly positive element a, e.g. $\sum_{n\in\mathbb{N}} 2^{-n}\sigma^{-n}(\varphi(a))$ respectively $\varphi(a)$.

E and $\mathcal{O}(\mathcal{H}(A, h))$ are stable if A is stable, because then $D = \varphi(A)$ is stable, $\mathcal{O}(\mathcal{H}(A, h)) = DED$ is stable (by Remark 2.4), and E is stable as the inductive limit of the stable hereditary C*-subalgebras $U^{-n}DEDU^n$ ($n = 1, 2, \ldots$), cf. [10].

Thus $\mathcal{O}(\mathcal{H}(A, h))$ is isomorphic to $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ by Lemma 2.13 if A is stable and contains a strictly positive element. □

Corollary 3.7.

$$\mathcal{O}(\mathcal{H}(A, h)) \otimes \mathbb{K} \cong ([D]_\sigma \rtimes_\sigma \mathbb{Z}) \otimes \mathbb{K}$$

if A is separable and $h\colon A \to \mathcal{M}(A)$ is non-degenerate. Where $D := h_{1,\infty}(A)$.

Proof. By the proof of Proposition 3.6, $\mathcal{O}(\mathcal{H}(A,h)) \otimes \mathbb{K}$ is a full hereditary C*-subalgebra of $([D]_\sigma \rtimes_\sigma \mathbb{Z}) \otimes \mathbb{K}$, and both algebras are separable. The algebras are isomorphic by Lemma 2.13. $\square$

4 Ideals of certain crossed products

Throughout this section we assume that σ is an automorphism of a C*-algebra B and that D is a C*-subalgebra of B.

We call a C*-subalgebra $E \subset B$ *σ-invariant* if $\sigma(E) = E$. In this case we will denote the restriction of σ to E again by σ. We also write σ for the second adjoint $\sigma^{**} \in \mathrm{Aut}(B^{**})$ of σ. (It can not cause confusions by Proposition 4.4.)

Definition 4.1. For a C*-subalgebra A of a C*-algebra B the *normalizer* of A in B is defined by

$$\mathcal{N}(B,A) := \{b \in B : \ bA + Ab \subset A\},$$

and the *annihilator* of A in B by

$$\mathrm{Ann}(B,A) := \{b \in B : \ bA = 0 = Ab\}\,.$$

We write also $\mathcal{N}(A)$ (respectively $\mathrm{Ann}(A)$) instead of $\mathcal{N}(B,A)$ (respectively $\mathrm{Ann}(B,A)$) if no confusion can arise.

4.1 Identification of some crossed product

Definition 4.2. Let $\sigma \in \mathrm{Aut}(B)$ and $D \subset B$ a C*-subalgebra. We say that D is *σ-modular* if D has following properties (α), (β) and (γ).

(α) $D\sigma(D) = \sigma(D)$,

(β) $\mathrm{Ann}(D + \sigma(D), \sigma(D)) = \{0\}$,

(γ) $D \cap \sigma(D) = \{0\}$.

For $D \subset B$, let $[D]_\sigma$ denote the smallest σ-invariant C*-subalgebra of B containing D.

Remark 4.3. (i) Property (α) means equivalently that $\sigma(D)$ is the closure of span $(D\sigma(D))$, i.e. $D\sigma(D) \subset \sigma(D)$ and every approximate unit of D is also an (outer) approximate unit for $\sigma(D)$. (Special case of the Cohen factorization theorem, cf. [5], [2, thm. I, §11.10].)
(ii) Note that (α) implies $D\sigma^k(D) = \sigma^k(D)$ for $k \in \mathbb{N}$ by induction, because $D\sigma^{k+1}(D) = D\sigma^k(D\sigma(D)) = (D\sigma^k(D))\sigma^{k+1}(D)$. (Below we show that $[D]_\sigma$ is the closure $D_{-\infty,\infty}$ of $\sum_{k\in\mathbb{Z}} \sigma^k(D)$ if $D\sigma^k(D) \subset \sigma^k(D)$ for $k \in \mathbb{N}$.)
(iii) Under assumption (α) the assumption (β) means that $\sigma(D)$ is an essential ideal of $D + \sigma(D)$.
(iv) Property (γ) equivalently means that $a + \sigma(b) = 0$ implies $a = b = 0$ for $a, b \in D$. Thus, under assumption (γ), the algebra D satisfies (β) if and only if $a, b \in D$ and $(a + \sigma(b))\,\sigma(D) = \{0\}$ together imply that $a = 0$.

In this section we prove the following propositions.

Proposition 4.4. *Suppose that $D \subset B$ and $\sigma \in \mathrm{Aut}\,(B)$ satisfy properties (α) and (γ) of Definition 4.2, that $D_1 \subset B_1$ and $\sigma_1 \in \mathrm{Aut}\,(B_1)$ satisfy $D_1\sigma_1(D_1) = D_1$, and that $\varphi\colon D \to D_1$ is a *-homomorphism with $\varphi\left(\sigma^{-1}(a)b\right) = \sigma_1^{-1}\left(\varphi(a)\right)\varphi(b)$ for all $a, b \in D$. Then φ extends to a *-homomorphism $\varphi_e\colon [D]_\sigma \to [D_1]_{\sigma_1}$ with $\varphi_e\left(\sigma(c)\right) = \sigma_1\left(\varphi_e(c)\right)$ for $c \in [D]_\sigma$.*

*If D is σ-modular in the sense of Definition 4.2 and if φ is a *-monomorphism (respectively a *-isomorphism from D onto D_1) then φ_e is a monomorphism (respectively φ_e is an isomorphism from $[D]_\sigma$ onto $[D_1]_{\sigma_1}$).*

Proposition 4.5. *Suppose that $\sigma \in \mathrm{Aut}\,(B)$ and $D \subset B$ is σ-modular in the sense of Definition 4.2. Every non-zero closed ideal I of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ has non-zero intersection $J = I \cap D$ with D, and J satisfies $J\sigma(D) \subset \sigma(J)$.*

In particular, if $\varrho\colon [D]_\sigma \rtimes_\sigma \mathbb{Z} \to \mathcal{L}\,(H)$ is a *-representation with $\varrho|D$ faithful, then ϱ is faithful.

Corollary 4.6. *Suppose $\sigma \in \mathrm{Aut}\,(B)$ and that $D \subset B$ is σ-modular (cf. Definition 4.2). If $J\sigma(D) \not\subset \sigma(J)$ holds for every closed non-trivial ideal J of D then $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ is a simple* C**-algebra.*

We need some notation for the proofs of Propositions 4.4 and 4.5. For

$m \le n$ and any C*-subalgebra $D \subset B$ let:

$$\begin{aligned} D_{m,n} &:= \sigma^{m-1}(D) + \cdots + \sigma^{n-1}(D)\,, \\ D_{-\infty,\infty} &:= \overline{\bigcup_{k\in\mathbb{N}} D_{-k,k}}\,, \end{aligned} \tag{4.1}$$

$$D_{-\infty,n} := \overline{\bigcup_{k\in\mathbb{N}} D_{n-k,n}}\,, \qquad D_{n,\infty} := \overline{\bigcup_{k\in\mathbb{N}} D_{n,n+k}}\,.$$

It will be shown in Lemma 4.9 that these vector spaces are C*-subalgebras of B under the additional assumption

$$D\sigma^k(D) \subset \sigma^k(D) \qquad \text{for} \quad k \in \mathbb{N}. \tag{4.2}$$

Lemma 4.7. *Under the assumption (4.2) holds:*

(i) $\sigma^j(D_{m,n}) = D_{m+j,n+j}$ *for* $j, m, n \in \mathbb{Z}$, $m \le n$.

(ii) $D_{m,n}D_{i,j} \subset D_{\max(m,i),\max(n,j)}$ *for* $m \le n$ *and* $i \le j$, $i, j, m, n \in \mathbb{Z}$.

Proof. (i) is obvious from definition.

Ad (ii): $\sigma^i(D)\sigma^j(D) = \sigma^i\left(D\sigma^{j-i}(D)\right)$ for $i \le j$. Since $\left(\sigma^i(D)\sigma^j(D)\right)^* = \sigma^j(D)\sigma^i(D)$, we get $\sigma^i(D)\sigma^j(D) \subset \sigma^{\max(i,j)}(D)$ for all $i, j \in \mathbb{Z}$.

This implies (ii) by definition of $D_{m,n}$. □

Lemma 4.8. *Let E, F be* C*-*subalgebras of B. If $EF \subset F$ then the sum $E + F$ is a* C*-*subalgebra of B.*

Proof. $E + F$ is a *-subalgebra, F is a closed ideal of $E + F$ and $E \cap F$ is a closed ideal in E, because $FE = F^*E^* = (EF)^* \subset F^* = F$. The map

$$\varphi\colon c + (E \cap F) \in E/(E \cap F) \mapsto c + F \in (E + F)/F$$

is a well-defined *-isomorphism from $E/(E\cap F)$ onto $(E+F)/F$. This shows, that $(E + F)/F \subset \mathcal{N}(B, F)/F$ is a C*-algebra. Therefore the pre-image $E + F$ of the quotient map is closed. □

Lemma 4.9. *Let $D\sigma^k(D) \subset \sigma^k(D)$ for $k \in \mathbb{N}$. Then, for $m, n \in \mathbb{Z}$ with $m \le n$:*

(i) $D_{m,n}$ is a C*-*subalgebra of B and $D_{i,n}$ is an ideal of $D_{m,n}$ for $m \le i \le n$.*

(ii) $D_{-\infty,\infty}$ *is the smallest* σ*-invariant* C*-*subalgebra* $[D]_\sigma$ *of* B *containing* D*, and is the inductive limit of the* C*-*algebras* $(D_{-k,k})_{k\in\mathbb{N}}$.

If, in addition, D *is of type I, then* $[D]_\sigma$ *is the inductive limit of type I algebras.*

(iii) $D_{-\infty,n}$ *is a* C*-*algebra and satisfies* $\sigma^{-1}(D_{-\infty,n}) \subset D_{-\infty,n}$.

(iv) $D_{n,\infty}$ *is a closed ideal of* $[D]_\sigma$ *and satisfies* $\sigma(D_{n,\infty}) \subset D_{n,\infty}$.

Proof. (i) follows from Lemma 4.7 and by induction over $n \geq m$ by applying Lemma 4.8 to $D_{m,n+1} = D_{m,n} + \sigma^n(D)$.

Ad (ii): It is obvious from the definition of $D_{-\infty,\infty}$ and (i) that it is the inductive limit of the sequence $D_{-1,1} \subset D_{-2,2} \subset \ldots$ of C*-algebras. The σ-invariance follows from Lemma 4.7(i). Clearly $D_{-\infty,\infty}$ is the smallest σ-invariant closed vector subspace of B containing D.

By (i), $D_{-k,k}$ has an ideal decomposition series

$$\sigma^{k-1}(D) = D_{k,k} \subset D_{k-1,k} \subset \cdots \subset D_{k-j,k} \subset \cdots \subset D_{-k+1,k} \subset D_{-k,k}\,.$$

The intermediate factors are isomorphic to $D/(D \cap D_{2,j+1})$, which are of type I if D is type I. Thus, $D_{-k,k}$ is of type I if D is of type I in addition to (4.2).

Ad (iii),(iv): In the same way one gets that $D_{-\infty,n}$ (respectively $D_{n,\infty}$) is the inductive limit of $D_{-k,n}$ (respectively of $D_{n,k}$) for $k \to \infty$. Observe $\sigma^{-1}(D_{-\infty,n}) = D_{-\infty,n-1} \subset D_{-\infty,n}$ and $\sigma(D_{n,\infty}) = D_{n+1,\infty} \subset D_{n,\infty}$.

$D_{n,\infty}$ is a closed ideal of $D_{-\infty,\infty}$ by Lemma 4.7(ii). □

Lemma 4.10. *If* $D\sigma^k(D) \subset \sigma^k(D)$ *for* $k \in \mathbb{N}$, $D \cap \sigma(D) = 0$, *and if* $\sigma(D)$ *is an essential ideal of* $D + \sigma(D)$ *then* $D_{j,n}$ *is an essential ideal of* $D_{m,n}$ *for all* $m \leq j \leq n$, $j, n \in \mathbb{Z}$, $m = -\infty$ *or* $m \in \mathbb{Z}$.

If I *is a non-zero* σ*-invariant closed ideal of* $[D]_\sigma$*, then* $J := I \cap D$ *is non-zero and satisfies* $J\sigma^k(D) \subset \sigma^k(D)$ *for* $k \in \mathbb{N}$.

Proof. First we show that the ideal $D_{n,n}$ of $D_{1,n}$ is essential by induction: $n = 1$ is trivial and $n = 2$ is true by assumption. Suppose it is true for n. Let $x \in D_{1,n+1}$ with $x \cdot D_{n+1,n+1} = 0$. x can be written as the sum of $y \in D_{1,n}$ and $z \in D_{n+1,n+1}$. $ay \in D_{n,n}$ and $az \in D_{n+1,n+1}$ for all $a \in D_{n,n}$, i.e. $ax \in D_{n,n} + D_{n+1,n+1} = \sigma^{n-1}(D_{1,2})$. But $D_{n+1,n+1} = \sigma^{n-1}(D_{2,2})$ is essential in $\sigma^{n-1}(D_{1,2})$ by assumption, thus $ax \cdot D_{n+1,n+1} = 0$ implies $ax = 0$.

Then $ay = -az \in D_{n,n} \cap D_{n+1,n+1} = \{0\}$ shows $ay = az = 0$, but $D_{n,n}$ was assumed essential in $D_{1,n}$ implying $y = 0$ and $x = z$ and this again implies $x = 0$.

By applying σ^{1-m} to $D_{m,n}$, we see that $D_{n,n}$ is essential in $D_{m,n}$. $D_{j,n}$ is an ideal of $D_{m,n}$ by Lemma 4.9(i). $D_{j,n}$ is essential, because $D_{j,n} \supset D_{n,n}$.

It follows that $D_{j,n}$ is an essential ideal of $D_{-\infty,n} = \mathrm{indlim}_{m\to-\infty} D_{m,n}$. Indeed, $D_{j,n}$ is a closed ideal of $D_{-\infty,n}$, and the natural *-homomorphism from $D_{-\infty,n}$ to $\mathcal{M}(D_{j,n})$ is isometric on $\bigcup_{m\leq j} D_{m,n} \subset D_{-\infty,n}$, thus is faithful on $D_{-\infty,n}$.

Since $D_{-\infty,\infty}$ is the inductive limit of $D_{-k,k}$ there exists $k \in \mathbb{N}$ with $I \cap D_{-k,k} \neq \{0\}$ if I is non-zero. If I is non-zero and σ-invariant, then there exists $n = 2k+1 \in \mathbb{N}$ with non-zero intersection $D_{1,n} \cap I$. Since $D_{n,n}$ is an essential ideal of $D_{1,n}$, the intersection $D_{n,n} \cap I = \sigma^{n-1}(D \cap I)$ is non-zero.

Let $J := I \cap D$ then $J\sigma^k(D) \subset \sigma^k(J)$ for $k \in \mathbb{N}$, because $J\sigma^k(D) \subset I$, $J\sigma^k(D) \subset \sigma^k(D)$ and $I \cap \sigma^k(D) = \sigma^k(J)$. □

Lemma 4.11. *Suppose $D\sigma^k(D) \subset \sigma^k(D)$ for $k \in \mathbb{N}$ and $D \cap \sigma(D) = \{0\}$. If $\sigma(D)$ is essential in $D + \sigma(D)$ or if $D\sigma(D) = \sigma(D)$, then $D_{m,n}$ is as a vector space the direct sum of the subspaces $D_{j,j} = \sigma^{j-1}(D) \cong D$ for $m \leq j \leq n$.*

In particular, $D_{m,n} = D_{m,j} + D_{j+1,n}$ and $D_{m,j} \cap D_{j+1,n} = 0$ for $m \leq j < n$.

Proof. It suffices to check the case $m = 1$ by induction over n, because $D_{m,n} = \sigma^{m-1}(D_{1,n-m+1})$. For $n = 1$ this is trivial and for $n = 2$ it follows from $D \cap \sigma(D) = \{0\}$. Let $d_j \in D_{j,j}$ for $j = 1, \ldots, n+1$ with $d_1 + \cdots + d_n + d_{n+1} = 0$. If $d_1 + \cdots + d_n = 0$ we get by induction assumption that $d_1 = d_2 = \cdots = d_n = 0$ and thus $d_{n+1} = 0$. Suppose that $d_1 + \cdots + d_n \neq 0$. Then $d_{n+1} \neq 0$. If $\sigma(D)$ is essential in $D + \sigma(D)$, then $D_{n,n} = \sigma^{n-1}(D)$ is essential in $D_{1,n}$ by Lemma 4.10. Thus there is $e \in D_{n,n}$ with $f := \sigma^{1-n}((d_1 + \cdots + d_n) \cdot e) \neq 0$, and $g := \sigma^{1-n}(d_{n+1}e) \neq 0$ and $f = -g \in D \cap \sigma(D)$ which contradicts $D \cap \sigma(D) = \{0\}$.

If $D\sigma(D) = \sigma(D)$ then there exists $e \in D_{n,n}$ with $ed_{n+1} \neq 0$, because

$$D_{n,n} \cdot D_{n+1,n+1} = \sigma^{n-1}(D\sigma(D)) = \sigma^n(D) = D_{n+1,n+1}$$

and the latter implies that $D_{n,n}$ contains an approximate unit for $D_{n+1,n+1}$. With f and g as above, we get again $f = -g \in D \cap \sigma(D)$. □

Proof of Proposition 4.4. By induction over $n = j - i$ we get $\varphi(\sigma^{i-j}(f)g) = \sigma_1^{i-j}(\varphi(f))\varphi(g)$ for $i \leq j$, $f, g \in D$, indeed:

By assumption we find $f_1, g_1 \in D$ with $g = \sigma^{-1}(f_1)\, g_1$ because $D = \sigma^{-1}(D\sigma(D))$. The induction step uses

$$\varphi\left(\sigma^{-n-1}(f)g\right) = \varphi\left(\sigma^{-1}\left(\sigma^{-n}(f)f_1\right)g_1\right),$$

where $\sigma^{-n}(f)f_1$ is in D.

Thus

$$\varphi\left(\sigma^{1-j}\left(d_i d_j\right)\right) = \sigma_1^{i-j}\left(\varphi\left(\sigma^{1-i}\left(d_i\right)\right)\right)\varphi\left(\sigma^{1-j}\left(d_j\right)\right) \tag{4.3}$$

for $d_i \in D_{i,i}$, $d_j \in D_{j,j}$, and $i \leq j$.

Let $d_j \in D_{j,j}$ and $\varphi_e(d_j) := \sigma_1^{j-1}\varphi\left(\sigma^{1-j}(d_j)\right) \in (D_1)_{j,j}$ for $-k \leq j \leq k$, and let

$$\varphi_e\left(d_{-k} + d_{-k+1} + \cdots + d_k\right) := \varphi_e\left(d_{-k}\right) + \varphi_e\left(d_{-k+1}\right) + \cdots + \varphi_e\left(d_k\right).$$

Then φ_e is a well-defined linear map from $D_{-k,k}$ into $(D_1)_{-k,k}$ by Lemma 4.11, because $D \cap \sigma(D) = \emptyset$. φ_e is a *-homomorphism, because $\varphi_e(d_i d_j) = \varphi_e(d_i)\varphi_e(d_j)$ by Equation (4.3) and and $\varphi_e\left(d_j^*\right) = \varphi_e\left(d_j\right)^*$.

Since $[D]_\sigma = D_{-\infty,\infty}$ and $[D_1]_{\sigma_1} = (D_1)_{-\infty,\infty}$ are the inductive limits of the C*-algebras $D_{-k,k}$ respectively of $(D_1)_{-k,k}$, we get that φ_e uniquely extends to a *-homomorphism from $[D]_\sigma$ to $[D_1]_{\sigma_1}$. φ_e satisfies $\varphi_e\sigma = \sigma_1\varphi_e$ because $\varphi_e\left(\sigma^n(d)\right) = \sigma_1^n\left(\varphi(d)\right)$ for $n \in \mathbb{Z}$, $d \in D$.

If φ is a *-monomorphism from D into D_1, then the kernel of φ_e is a σ-invariant closed ideal I of $[D]_\sigma$ with $I \cap D = \{0\}$. It implies $I = \{0\}$ by Lemma 4.10 if D is σ-modular, i.e. if D satisfies (β) in addition.

In particular, φ_e is a *-monomorphism with dense image in $(D_1)_{-\infty,\infty} = [D_1]_{\sigma_1}$ if φ is an isomorphism from D onto D_1. □

Remark 4.12. Under the assumption of (α) and (γ), $\{D_{-k,1}\}$ is a decomposition series of ideals of $D_{-\infty,1}$ with intermediate factors isomorphic to D, and σ^{-1} defines an endomorphism of $D_{-\infty,1}$ such that $[D]_\sigma$ is the inductive limit of $\sigma^{-1}\colon D_{-\infty,1} \hookrightarrow D_{-\infty,1}$, i.e. $([D]_\sigma, \sigma^{-1})$ is the natural dynamical system corresponding to the endomorphism $\sigma^{-1}\colon D_{-\infty,1} \hookrightarrow D_{-\infty,1}$. In particular, $D_{-\infty,1}$ is of type I if A is of type I.

Indeed, the C*-algebras $D_{-k,1}$ are closed ideals of $D_{-n,1}$ for $n \geq k$ by Lemma 4.9(i). Since $D_{-\infty,1}$ is the inductive limit of $D_{-n,1}$ we get that $D_{-k,1}$ is also an ideal of $D_{-\infty,1}$. The quotient $D_{-(k+1),1}/D_{-k,1}$ is isomorphic to D by Lemma 4.11. σ^{-1} is an endomorphism of $D_{-\infty,1}$ because $\sigma^{-1}(D_{-\infty,1}) = D_{-\infty,0} \subset D_{-\infty,1}$. Clearly, the smallest σ-invariant C*-subalgebra of B containing $D_{-\infty,1}$ is the inductive limit of $D_{-\infty,1} \xrightarrow{\sigma^{-1}} D_{-\infty,1} \xrightarrow{\sigma^{-1}} D_{-\infty,1} \xrightarrow{\sigma^{-1}} \ldots$

and is the same as $D_{-\infty,\infty} = [D]_\sigma$. And σ^{-1} is the unique automorphism of $[D]_\sigma$ that induces σ^{-1} on $D_{-\infty,1}$.

Lemma 4.13. *Suppose $\sigma \in \mathrm{Aut}\,(B)$ and that $D \subset B$ is σ-modular. Let e_n denote the unit of $\overline{D_{n,n}}^{\mathrm{w}} \subset B^{**}$, then*

(i) $e_n \geq e_{n+1}$, $\sigma\,(e_n) = e_{n+1}$,

(ii) $([D]_\sigma)^{**} \cong \overline{D_{-\infty,\infty}}^{\mathrm{w}} \subset B^{**}$,

(iii) e_n *is the unit of* $(D_{n,\infty})^{**}$ *and is in the center of* $([D]_\sigma)^{**}$,

(iv) $\|a(e_{-n} - e_{n+1})\| = \|a\|$ *for* $a \in D_{-k,k}$ *and* $k, n \in \mathbb{N}$ *with* $k \leq n$.

Here $\overline{X}^{\mathrm{w}}$ denotes the ultra-weak closure of a subspace X of the W^*-algebra B^{**}. $\overline{X}^{\mathrm{w}}$ is naturally isomorphic to X^{**} if X is a closed subspace of B. We write also σ for the second conjugate σ^{**} of σ.

Proof. Ad (i): $\sigma\,(e_n) = e_{n+1}$, because $\sigma\,(D_{n,n}) = D_{n+1,n+1}$ and σ^{**} is weakly continuous on B^{**}. Since $D\sigma(D) = D$, $D_{n,n}$ contains an approximate unit for $D_{n+1,n+1}$, which implies $e_{n+1} \leq e_n$.

Ad (ii): (ii) follows from Lemma 4.9(ii).

Ad (iii): e_n is in $\overline{D_{n,\infty}}^{\mathrm{w}}$. If $a \in D_{k,k}$ and $k \geq n$, then $e_k \leq e_n$, $a \in e_k B^{**} e_k$, and $e_n a e_n = a$. Thus $e_n a e_n = a$ for $a \in D_{n,k}$, $k \geq n$, which implies $e_n b e_n = b$ for $b \in \overline{D_{n,\infty}}^{\mathrm{w}}$, i.e. e_n is the unit of $\overline{D_{n,\infty}}^{\mathrm{w}}$.

$D_{n,\infty}$ is a closed ideal of $[D]_\sigma \subset B$ by Remark 4.3(ii) and Lemma 4.9(iv), and e_n is the (open) support projection of $D_{n,\infty}$ in $\overline{D_{-\infty,\infty}}^{\mathrm{w}} \cong ([D]_\sigma)^{**}$. Thus e_n is in the center of $([D]_\sigma)^{**}$.

Ad (iv): The map $\rho\colon a \mapsto a\,(e_{-k} - e_{k+1})$ is a *-homomorphism from $D_{-k,k}$ into $\overline{D_{-k,k+1}}^{\mathrm{w}} \cong (D_{-k,k+1})^{**}$. If $\rho(a) = 0$ then $a = a e_{k+1}$. If $\{b_\tau\}$ is an approximate unit of D then $\sigma^k\,(b_\tau)$ tends weakly to e_{k+1}. By Hahn-Banach separation this implies that a is in the closed span of $D_{-k,k}\sigma^k(D) \subset \sigma^k(D)$. Since $D \cap \sigma(D) = \{0\}$, we get $D_{-k,k} \cap \sigma^k(D) = \{0\}$ by Lemma 4.11. Hence $a = 0$ and ρ is isometric on $D_{-k,k}$. Now $\|a(e_{-n} - e_{n+1})\| = \|a\|$ for $a \in D_{-k,k}$ follows from $(e_{-k} - e_{k+1})(e_{-n} - e_{n+1}) = e_{-k} - e_{k+1}$ for $0 \leq k \leq n$. □

We now return to the Proposition 4.5 stated at the beginning of this section:

Proof of Proposition 4.5. If I is a closed ideal of $[D]_\sigma \rtimes \mathbb{Z}$ then $I_1 := [D]_\sigma \cap I$ is a σ-invariant closed ideal of $[D]_\sigma$. The closed ideal $J := I \cap D = I_1 \cap D$ satisfies $J\sigma(D) \subset \sigma(J)$ by Lemma 4.10.

Suppose that $\varrho\colon [D]_\sigma \rtimes_\sigma \mathbb{Z} \to \mathcal{L}(H)$ is a *-representation of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ such that $\varrho|D$ is faithful, and let $I := \ker(\varrho)$. Then $I_1 := [D]_\sigma \cap I$ satisfies $I_1 \cap D = \{0\}$.

By Lemma 4.10 it follows that $I_1 = \{0\}$. Now Lemma 4.13 shows that Proposition A.29 applies. Hence, ϱ is faithful on $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ by Proposition A.29.

Equivalently: Every non-zero closed ideal I of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ has non-zero intersection $J = I \cap D$ with D. $\square$

4.2 Semi-invariant ideals

Definition 4.14. We say that a closed ideal J of D is (D,σ)-*semi-invariant*, if $J\sigma(D) \subset \sigma(J)$. J has (D,σ)-*cancellation* if $a, b \in D$ and $(a+\sigma(b))\sigma(D) \subset \sigma(J)$ implies that $a \in J$.

If J is semi-invariant and has cancellation, then $(a+\sigma(b))\sigma(D) \subset \sigma(J)$ implies that $a, b \in J$, because $bD \subset J$ implies $b \in J$.

Lemma 4.15. *Suppose that $D \subset B$ and $\sigma \in \mathrm{Aut}\,(B)$ satisfy conditions (α) and (γ) of Definition 4.2, and that J is a closed ideal of D with $J\sigma(D) \subset \sigma(J)$.*

(i) *$J\sigma^k(D) \subset \sigma^k(J)$ for $k \in \mathbb{N}$, and $[J]_\sigma$ is a closed σ-invariant ideal of $[D]_\sigma$.*

(ii) *$(D + \sigma(D)) \cap [J]_\sigma = J + \sigma(J)$.*

(iii) *The system $D_1 := (D + [J]_\sigma)/[J]_\sigma \subset B_1 := [D]_\sigma/[J]_\sigma$, $\sigma_1 \in \mathrm{Aut}\,(B_1)$ with $\sigma_1(a + [J]_\sigma) := \sigma(a) + [J]_\sigma$ satisfies again (α) and (γ) of Definition 4.2, i.e. $D_1\sigma_1(D_1) = D_1$ and $D_1 \cap \sigma_1(D_1) = \{0\}$.*

(iv) *D_1 and σ_1 satisfy also condition (β), i.e. $\mathrm{Ann}\,(D_1 + \sigma_1(D_1), \sigma_1(D_1)) = \{0\}$, if and only if J has (D,σ)-cancellation.*

(v) *If J has (D,σ)-cancellation, then $K = [J]_\sigma$ for every σ-invariant closed ideal K of $[D]_\sigma$ with $K \cap D = J$.*

Proof. First note that $J\sigma^k(J) \subset \sigma^k(J)$ and $J_{m,n}$ (respectively $J_{-\infty,\infty}$) is a closed ideal of of $D_{m,n}$ (respectively of $D_{-\infty,\infty}$), if J is a closed ideal of D with $J\sigma^k(D) \subset \sigma^k(J)$ for all $k \in \mathbb{N}$. This follows from Lemma 4.10(i) and $(\sigma^i(D)\sigma^j(J))^* = \sigma^j(J)\sigma^i(D) \subset \sigma^{\max(i,j)}(J)$ for $i, j \in \mathbb{Z}$: If $i \leq j$, then $\sigma^{j-i}(J)D = \sigma^{j-i}(JD)D = \sigma^{j-i}(J)\sigma^{j-i}(D)D \subset \sigma^{j-i}(J)\sigma^{j-i}(D) = \sigma^{j-i}(J)$, and, if $i \geq j$, then

$$\sigma^j(J)\sigma^i(D) = \sigma^j\left(J\sigma^{i-j}(D)\right) \subset \sigma^j\left(\sigma^{i-j}(J)\right) = \sigma^i(J)\,.$$

Ad (i): $J\sigma^{k+1}(D) = J\sigma^k(D)\sigma^{k+1}(D) \subset \sigma^k(J\sigma(D))$ by (α) and induction assumption. The right side is contained in $\sigma^{k+1}(J)$. Thus $[J]_\sigma = J_{-\infty,\infty}$ is a closed ideal of $[D]_\sigma = D_{-\infty,\infty}$.

Ad (ii): Above we have seen that $J_{-k,k}$ is a closed ideal of $D_{-k,k}$ for $k \in \mathbb{N}$. $[J]_\sigma = J_{-\infty,\infty}$ is the inductive limit of $\{J_{-k,k}\}_{k\in\mathbb{N}}$ by Lemma 4.9(ii). It follows $\operatorname{dist}(a, [J]_\sigma) = \lim_k \operatorname{dist}(a, J_{-k,k})$ and $\operatorname{dist}(a, J_{-k,k}) = \operatorname{dist}(a, (D + \sigma(D)) \cap J_{-k,k})$ for $a \in D + \sigma(D)$. But $(D + \sigma(D)) \cap J_{-k,k} = J + \sigma(J)$ by Lemma 4.11.

Ad (iii): Let $\pi\colon [D]_\sigma \to B_1 = [D]_\sigma/[J]_\sigma$ the quotient map. Then $\pi \circ \sigma = \sigma_1 \circ \pi$ and $\pi(D) = D_1$. Thus, $D_1\sigma_1(D_1) = \pi(D\sigma(D)) = D_1$ by (α). If $d, e \in D$, $d_1 = \pi(d)$ and $e_1 = \pi(e)$ satisfy $d_1 = \sigma_1(e_1)$, then $d - \sigma(e) \in [J]_\sigma$. There are $f, g \in J$ with $d - \sigma(e) = f + \sigma(g)$, by part (ii). $d - f = \sigma(e + g)$ implies $d = f$ and $d_1 = \pi(f) = 0$ by (γ) for (D, σ).

Ad (iv): $\sigma(D) \cap [J]_\sigma = \sigma(D) \cap (J + \sigma(J)) = \sigma(J)$ by (ii) and property (γ). Let $d, e \in D$. $(\pi(d) + \sigma_1(\pi(e)))\,\sigma_1(D_1) = \{0\}$ is equivalent to $(a + \sigma(e))\,\sigma(D) \subset \sigma(D) \cap [J]_\sigma$. Thus, $\operatorname{Ann}(D_1 + \sigma_1(D_1), \sigma_1(D_1)) = \{0\}$ if and only if J has (D, σ)-cancellation.

Ad (v): Clearly $[J]_\sigma \subset K$ if K is a σ-invariant closed ideal with $J = D \cap K$. Then $\pi(K)$ is a closed ideal of $B_1 = [D]_\sigma/[J]_\sigma$ with $\pi(K) \cap \pi(D) = \{0\}$. Since $D_1 \subset B_1$ is σ_1-modular, it follows that $\pi(K) = \{0\}$ by Lemma 4.10, i.e. $K = [J]_\sigma$. □

Proposition 4.16. *Suppose that $\sigma \in \operatorname{Aut}(B)$ and $D \subset B$ is σ-modular in the sense of Definition 4.2. If I is a closed ideal of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ such that $J := D \cap I$ has (D, σ)-cancellation (cf. Definition 4.14), then I is the natural image of $[J]_\sigma \rtimes_\sigma \mathbb{Z}$ in $[D]_\sigma \rtimes_\sigma \mathbb{Z}$.*

Proof. $K := [D]_\sigma \cap I$ is a σ-invariant ideal of $[D]_\sigma$. Thus $J = D \cap I$ satisfies $J\sigma(D) \subset \sigma(J)$ and $[J]_\sigma$ is a σ-invariant closed ideal of $[D]_\sigma$. Let $D_1 := (D + [J]_\sigma)/[J]_\sigma$, $B_1 := [D]_\sigma/[J]_\sigma$, and $\sigma_1 \in \operatorname{Aut}(B_1)$ defined by

$\sigma_1(a+[J]_\sigma) := \sigma(a)+[J]_\sigma$. The natural image K of $[J]_\sigma \rtimes_\sigma \mathbb{Z}$ in $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ is contained in I and is the kernel of the natural epimorphism $\pi \rtimes \mathbb{Z}$ from $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ onto $[D_1]_{\sigma_1} \rtimes_{\sigma_1} \mathbb{Z}$ (cf. Remark A.25(vi)). Let $I_1 := (\pi \rtimes \mathbb{Z})(I)$. I_1 is a closed ideal of $[D_1]_{\sigma_1} \rtimes_{\sigma_1} \mathbb{Z}$ with $I_1 \cap D_1 = \{0\}$. $D_1 \subset B_1$ is σ_1-modular by Lemma 4.15(iv), because J is (D,σ)-semi-invariant and has (D,σ)-cancellation. Thus, $I_1 = \{0\}$ by Proposition 4.5, i.e. $I = K$. □

Remark 4.17. In the above proof we have also shown:

If K is a σ-invariant closed ideal of $D_{-\infty,\infty}$ such that $J := D \cap K$ has (D,σ)-cancellation, then $K = J_{-\infty,\infty}$ and every *-representation

$$\varrho\colon D_{-\infty,\infty} \rtimes_\sigma \mathbb{Z} \to \mathcal{L}(H)$$

with $\ker(\varrho|D) = D \cap K$ has kernel $K \rtimes_\sigma \mathbb{Z}$.

Thus, ϱ defines a faithful representation of $(D_{-\infty,\infty}/K) \rtimes_{[\sigma]} \mathbb{Z}$, where $[\sigma]$ is induced by σ on $D_{-\infty,\infty}/K$.

Corollary 4.18. *Suppose that $\sigma \in \mathrm{Aut}(B)$ and $D \subset B$ is σ-modular (cf. Definition 4.2).*

If every closed ideal J of D with $J\sigma(D) \subset \sigma(J)$ has (D,σ)-cancellation (cf. Definition 4.14), then the map

$$I \in \mathbb{I}([D]_\sigma \rtimes_\sigma \mathbb{Z}) \mapsto I \cap D \in \mathbb{I}(D),$$

is a lattice isomorphism from $\mathbb{I}([D]_\sigma \rtimes_\sigma \mathbb{Z}) \cong \mathbb{O}(\mathrm{Prim}([D]_\sigma \rtimes_\sigma \mathbb{Z}))$ onto the lattice $\mathbb{I}(D)^\sigma$ of closed ideals J in D with $J\sigma(D) \subset \sigma(J)$.

The inverse lattice isomorphism is given by $J \mapsto [J]_\sigma \rtimes_\sigma \mathbb{Z}$.

D is a regular C*-*subalgebra of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ in the sense of Definition 1.2.*

(Here we identify $[J]_\sigma \rtimes_\sigma \mathbb{Z}$ with its natural image in $[D]_\sigma \rtimes_\sigma \mathbb{Z}$.)

Proof. We show that

$$J \mapsto [J]_\sigma \mapsto [J]_\sigma \rtimes_\sigma \mathbb{Z}$$

define a lattice isomorphisms from the lattice $\mathbb{I}(D)^\sigma$ of (D,σ)-semi-invariant closed ideals of D onto the lattice $\mathbb{I}([D]_\sigma)^\sigma$ of σ-invariant closed ideals of $[D]_\sigma$, and from $\mathbb{I}([D]_\sigma)^\sigma$ onto $\mathbb{I}([D]_\sigma \rtimes_\sigma \mathbb{Z})$, and that the map

$$I \in \mathbb{I}([D]_\sigma \rtimes_\sigma \mathbb{Z}) \mapsto D \cap I \in \mathbb{I}(D)^\sigma$$

is the inverse of $J \in \mathbb{I}(D)^\sigma \mapsto [J]_\sigma \rtimes_\sigma \mathbb{Z}$.

If J is a closed ideal with $J\sigma(D) \subset \sigma(J)$, then $[J]_\sigma$ is a σ-invariant closed ideal of $[D]_\sigma$ by Lemma 4.15(i). Thus $[J]_\sigma \rtimes_\sigma \mathbb{Z}$ is a closed ideal of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$. Clearly, the maps $J \in \mathbb{I}\,(D)^\sigma \mapsto [J]_\sigma$ and $K \in \mathbb{I}\,([D]_\sigma)^\sigma \mapsto K \rtimes_\sigma \mathbb{Z}$ are order preserving (i.e. "$\subset$"-preserving). $J \mapsto [J]_\sigma$ is injective, because by Lemma 4.15(ii) and (γ) of Definition 4.2: $J \subset [J]_\sigma \cap D \subset (J + \sigma(J)) \cap D \subset J$.

If K is a σ-invariant closed ideal of $[D]_\sigma$, then $K = [D]_\sigma \cap (K \rtimes_\sigma \mathbb{Z})$ by Remark A.25(iii). In particular, $K \mapsto K \rtimes_\sigma \mathbb{Z}$ is an injective map from the lattice of σ-invariant closed ideals of $[D]_\sigma$ into the lattice of closed ideals of $[D]_\sigma \rtimes \mathbb{Z}$.

If I is a closed ideal of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$, then $J := D \cap I$ is a closed ideal with $J\sigma(D) \subset \sigma(J)$. By Proposition 4.16, $I = [D \cap I]_\sigma \rtimes_\sigma \mathbb{Z}$, because J has (D,σ)-cancellation (by assumption).

Hence, $I \in \mathbb{I}\,([D]_\sigma \rtimes_\sigma \mathbb{Z}) \mapsto D \cap I \in \mathbb{I}\,(D)^\sigma$ is an order isomorphism which is invertible with inverse map $J \mapsto [J]_\sigma \rtimes \mathbb{Z}$. It follows that $J \mapsto [J]_\sigma$ is injective. If the superposition of two injective maps is surjective, then the maps are both invertible.

If I_1, I_2 are closed ideals of $[D]_\sigma \rtimes \mathbb{Z}$, then $J_1 := I_1 \cap D$, $J_2 := I_2 \cap D$ and $J_3 := J_1 + J_2$ are (D,σ)-semi-invariant.

$I_1 + I_2$ is the l.u.b. of $\{I_1, I_2\}$ in the lattice $\mathbb{I}\,([D]_\sigma \rtimes \mathbb{Z})$, and J_3 is the l.u.b. of $\{J_1, J_2\}$ in the lattice $\mathbb{I}\,(D)^\sigma$. Thus $D \cap (I_1 + I_2) = (D \cap I_1) + (D \cap I_2)$. It follows that D is a regular C*-subalgebra of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$. □

Remark 4.19. Suppose that $D \subset B$ and $\sigma \in \mathrm{Aut}\,(B)$ satisfies condition (α). Define a natural *-homomorphism $h\colon D \to \mathcal{M}\,(D)$ by $h(a)b := \sigma^{-1}(a)b$. Clearly, h is non-degenerate (i.e. $h(D)D = D$) and the kernel of h is equal to $D \cap \mathrm{Ann}\,(D + \sigma(D), \sigma(D))$.

Then the conditions (β) and (γ), i.e. $\mathrm{Ann}\,(D + \sigma(D), \sigma(D)) = \{0\}$ and $D \cap \sigma(D) = \{0\}$, are equivalent to $h^{-1}\,(h(D) \cap D) = 0$, i.e., $h\colon D \to \mathcal{M}\,(D)$ is faithful and $h(D) \cap D = \{0\}$.

(Indeed: $h^{-1}(h(D) \cap D) = \{a \in D:\ h(a) \in D\}$ is the set of all $a \in D$ with the property that there is $b \in D$ with $\sigma^{-1}\,((a - \sigma(b))\sigma(D)) = (h(a) - b)D = \{0\}$, i.e. with $a + \sigma(-b) \in \mathrm{Ann}\,(D + \sigma(D), \sigma(D))$.)

A closed ideal J of D satisfies $J\sigma(D) \subset \sigma(J)$ if and only if $h(J)D \subset J$, i.e. , $h(J) \subset \mathcal{M}\,(D, J)$.

J has (D,σ)-cancellation in the sense of Definition 4.14, if and only if, the existence of $b \in D$ with $(\sigma^{-1}(a) + b)\, D \subset J$ implies $a \in J$. Thus, J has

(D,σ)-cancellation, if and only if,

$$h^{-1}\left(h(D)\cap\left(D+\mathcal{M}\left(D,J\right)\right)\right)\subset J\,.$$

Hence, a closed ideal J of D is (D,σ)-semi-invariant and has (D,σ)-cancellation if and only if $h(D)\cap(D+\mathcal{M}\left(D,J\right))=h(J)$.

Corollary 4.20. *Suppose that $\sigma\in\mathrm{Aut}\,(B)$ and that $D\subset B$ is σ-modular in the sense of Definition 4.2. Let $h\colon D\to\mathcal{M}\,(D)$ denote the non-degenerate *-monomorphism given by $h(a)b:=\sigma^{-1}(a)b$ for $a,b\in D$.*

Then the hereditary C^*-subalgebra of $[D]_\sigma\rtimes_\sigma\mathbb{Z}$ generated by D is full in $[D]_\sigma\rtimes_\sigma\mathbb{Z}$ and is isomorphic to the Cuntz–Pimsner algebra $\mathcal{O}\left(\mathcal{H}(D,h)\right)$.

If every closed ideal J of D with $h(J)D\subset J$ satisfies

$$h(J)=h(D)\cap\left(D+\mathcal{M}\left(D,J\right)\right),$$

then $I\mapsto J:=D\cap I$ defines an isomorphism from $\mathbb{I}\left(\mathcal{O}\left(\mathcal{H}(D,h)\right)\right)$ onto the lattice

$$\{J\in\mathbb{I}\,(D)\;:\;\;h(J)D\subset J\}\,.$$

Proof. By Remark 4.19 and Proposition 3.6, there exists a *-monomorphism φ from D into $\mathcal{O}\left(\mathcal{H}(D,h)\right)\subset Q^s\left(\mathcal{M}\left(D\right)\right)$ and a unitary U in the stable corona $Q^s\left(\mathcal{M}\left(D\right)\right):=\mathcal{M}\left(\mathcal{M}\left(D\right)\otimes\mathbb{K}\right)/\left(\mathcal{M}\left(D\right)\otimes\mathbb{K}\right)$ of $\mathcal{M}\left(D\right)$ such that $U\varphi(D)$ generates $\mathcal{O}\left(\mathcal{H}(D,h)\right)$ and that $\varphi\left(\sigma^{-1}(a)b\right)=U^*\varphi(a)U\varphi(b)$ for all $a,b\in D$. Moreover, the C^*-algebra E generated by $\bigcup_{n\in\mathbb{N}}U^{-n}\varphi(D)$ is naturally isomorphic to $[\varphi(D)]_{\sigma_1}\rtimes_{\sigma_1}\mathbb{Z}$ for $\sigma_1(b):=UbU^*$, and $\mathcal{O}\left(\mathcal{H}(D,h)\right)$ is the hereditary C^*-subalgebra of E generated by $\varphi(D)$.

By Proposition 4.4 it follows that φ extends to a *-monomorphism φ_e from $[D]_\sigma$ into $E\subset Q^s\left(\mathcal{M}\left(D\right)\right)$ with $\varphi_e\left(\sigma(a)\right)=U\varphi_e(a)U^*$ for $a\in[D]_\sigma$. By Remark A.25(vi) there is a *-homomorphism ϱ from $[D]_\sigma\times_\sigma\mathbb{Z}$ onto E with $\varrho|[D]_\sigma=\varphi_e$. ϱ is an isomorphism by Proposition 4.5 because $\varrho|D$ is faithful. ϱ maps the full hereditary C^*-subalgebra of $[D]_\sigma\times_\sigma\mathbb{Z}$ generated by D onto $\mathcal{O}\left(\mathcal{H}(D,h)\right)$, thus, the restriction of ϱ to $D([D]_\sigma\times_\sigma\mathbb{Z})D$ is an isomorphism onto $\mathcal{O}\left(\mathcal{H}(D,h)\right)=\varphi(D)E\varphi(D)$.

The lattice isomorphisms follow from Corollary 4.18 by Remark 4.19. $\square$

Corollary 4.21. *Suppose that $D\subset B$ and $\sigma\in\mathrm{Aut}\,(B)$ satisfy $D\sigma(D)=\sigma(D)$. If for the (non-degenerate) *-homomorphism $h\colon D\to\mathcal{M}\,(D)$ defined by $h(a)b:=\sigma^{-1}(a)b$ holds $h^{-1}\left(h(D)\cap D\right)=\{0\}$ and $h(J)D\not\subset J$ for every* *non-trivial closed ideal J of D, then $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ is simple and contains a copy of the Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H}(D,h))$ as a hereditary* C*-*subalgebra.*

If, in addition, D is σ-unital and stable, then $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ is isomorphic to $\mathcal{O}(\mathcal{H}(D,h))$.

4.3 (D,σ) and $\mathcal{H}(A,h)$

Here we translate the results on σ-modular algebras D in subsections 4.2 and 4.1 to the situation of our special case of Cuntz–Pimsner algebras.

4.22. We consider in this section the following situation as an example of the previous section:
Suppose that A is a C*-algebra and $h\colon A \to \mathcal{M}(A)$ a *-homomorphism with

(ND) h is *non-degenerate*, i.e. $h(A)A = A$,

(GP) $h(A)$ is in *general position*, i.e. $h(A) \cap A = 0$,

(FF) h is *faithful*, i.e. $\ker(h) = \{0\}$.

Then $h(A) + A$ is a C*-subalgebra of $\mathcal{M}(A)$ and A is an essential ideal of $h(A) + A$, because A is essential in $\mathcal{M}(A)$.

Since h is non-degenerate, h extends uniquely to a *-homomorphism $\mathcal{M}(h) : \mathcal{M}(A) \to \mathcal{M}(A)$ with $\mathcal{M}(h)(a)h(b)c = h(ab)c$ for $a \in \mathcal{M}(A)$ and $b, c \in A$. We denote $\mathcal{M}(h)$ also by h to keep notation simple. It follows from properties (ND) and (FF) that $h\colon \mathcal{M}(A) \to \mathcal{M}(A)$ becomes a strictly continuous *unital *-monomorphism.*

Let $B := \ell_\infty(\mathcal{M}(A))/c_0(\mathcal{M}(A))$ and $\sigma \in \mathrm{Aut}(B)$ be the automorphism induced by the forward-shift $(a_1, a_2, \dots) \mapsto (0, a_1, a_2, \dots)$ on $\ell_\infty(\mathcal{M}(A))$.

Now consider the natural embedding of the inductive limit G of

$$\mathcal{M}(A) \xrightarrow{h} \mathcal{M}(A) \xrightarrow{h} \cdots$$

into B, given by the unital **-monomorphisms* $h_{n,\infty}\colon \mathcal{M}(A) \to B$ with:

$$\begin{aligned} h^\infty\colon a \in \mathcal{M}(A) &\mapsto \big(a, h(a), h^2(a), \dots\big) \in \ell_\infty(\mathcal{M}(A)), \\ h_{n,\infty}(a) &:= \sigma^{n-1}\left(h^\infty(a) + c_0(\mathcal{M}(A))\right), \end{aligned}$$

compare Remark 3.5 and the notations in (i)–(vi) above Lemma A.27.

$$\begin{aligned} G &:= \operatorname{indlim}(h\colon \mathcal{M}(A) \to \mathcal{M}(A)) = \overline{\bigcup h_{k,\infty}(\mathcal{M}(A))} \\ &\subset B := \ell_\infty(\mathcal{M}(A))/c_0(\mathcal{M}(A)). \end{aligned}$$

σ induces an automorphism on G with

$$\sigma\left(h_{k,\infty}\left(\mathcal{M}\left(A\right)\right)\right) = h_{k+1,\infty}\left(\mathcal{M}\left(A\right)\right).$$

Then $h_{k,\infty}\colon \mathcal{M}\left(A\right) \xrightarrow{\sim} h_{k,\infty}(\mathcal{M}\left(A\right)) \subset h_{k+1,\infty}(\mathcal{M}\left(A\right)) \subset B$ define unital *-monomorphisms with

(HR) $\sigma h_{k,\infty} = h_{k+1,\infty}\,,\ h_{k+1,\infty} h = h_{k,\infty}\,,$ and $\sigma^{-1} h_{k,\infty} = h_{k,\infty} h$ for $k \in \mathbb{N}\,.$

Let $D := h_{1,\infty}(A)$. We have $h_{2,\infty}(h(A)+A) = h_{1,\infty}(A)+h_{2,\infty}(A) = D+\sigma(D)$, $\sigma^{n-1}(D) = h_{n,\infty}(A)$ for $n \in \mathbb{N}\,.$

The following Lemma 4.23 translates (ND), (GP), (FF) to the terminology in subsections 4.2 and 4.1.

Lemma 4.23. *With $D := h_{1,\infty}(A) \subset B$, $\sigma \in \mathrm{Aut}\,(B)$ as above we get from properties (ND), (GP) and (FF) of 4.22:*

(i) D is σ-modular in the sense of Definition 4.2.

(ii) $J \in \mathbb{I}\,(A)$ satisfies $h(J)A \subset J$, if and only if, $J_1 := h_{1,\infty}(J)$ is (D,σ)-semi-invariant in the sense of Definition 4.14.

(iii) $J \in \mathbb{I}\,(A)$ satisfies $h(J) = h(A) \cap (A + \mathcal{M}\,(A,J))$, if and only if, $J_1 := h_{1,\infty}(J)$ is (D,σ)-semi-invariant and has (D,σ)-cancellation in the sense of Definition 4.14.

Proof. Ad (i): $\sigma(D) = h_{2,\infty}(A)$ and $h_{2,\infty}h = h_{1,\infty}$ by (HR), thus:

Ad (α): $D\sigma(D) = h_{2,\infty}(h(A)A) = h_{2,\infty}(A) = \sigma(D)$.

Ad (β): $\sigma(D) = h_{2,\infty}(A)$ is essential in $D+\sigma(D) = h_{2,\infty}(h(A)+A) \subset h_{2,\infty}\left(\mathcal{M}\left(A\right)\right)$ because A is essential in $h(A)+A \subset \mathcal{M}\left(A\right)$ and $h_{2,\infty}$ is faithful on $\mathcal{M}\left(A\right)$.

Ad (γ): $D \cap \sigma(D) = h_{2,\infty}(h(A) \cap A) = \{0\}$.

Ad (ii): Since $h_{1,\infty}|A$ is a C*-isomorphism from A onto D, $J \mapsto J_1 := h_{1,\infty}(J)$ is a lattice isomorphism from $\mathbb{I}\,(A)$ onto $\mathbb{I}\,(D)$. It holds

$$\begin{aligned} J_1\sigma(D) &= h_{1,\infty}(J)h_{2,\infty}(A) = h_{2,\infty}(h(J)A) \\ \sigma(J_1) &= \sigma\left(h_{1,\infty}(J)\right) = h_{2,\infty}(J) \quad \text{and} \\ h^{-1}\left(h(A) \cap \mathcal{M}\left(A,J\right)\right) &= \{a \in A:\ h(a)A \subset J\}\,. \end{aligned}$$

Therefore, the following equivalences hold:

$$J_1\sigma(D) \subset \sigma\left(J_1\right) \Longleftrightarrow h(J)A \subset J \Longleftrightarrow J \subset h^{-1}\left(h(A) \cap \mathcal{M}\left(A,J\right)\right).$$

Ad (iii): $a \in A$ satisfies $h(a) \in \mathcal{M}(A, J) + A$, if and only if, there is $b \in A$ with $(h(a) + b)A \subset J$. If we apply $h_{2,\infty} = \sigma \circ h_{1,\infty}$, we see that $(h(a)+b)A \subset J$ is equivalent to $(a_1 + \sigma(b_1))\sigma(D) \subset \sigma(J_1)$ for $J_1 := h_{1,\infty}(J)$, $a_1 := h_{1,\infty}(a)$ and $b_1 := h_{1,\infty}(b)$. Hence, J_1 has (D, σ)-cancellation, if and only if, $h^{-1}(h(A) \cap (A + \mathcal{M}(A, J)) \subset J$. □

Remark 4.24. Conversely, if B_1 is a C*-algebra, $\sigma_1 \in \mathrm{Aut}(B_1)$ and if $A \subset B_1$ is σ-modular in the sense of Definition 4.2 (i.e. satisfies (α), (β) and (γ) for A in place of D), then $h \colon A \to \mathcal{M}(A)$ defined by $h(a)b := \sigma^{-1}(a)b$ has properties (ND), (GP) and (FF). The map $\varphi := h_{1,\infty}$ is a *-isomorphism from A onto D with $\varphi(\sigma_1^{-1}(a)b) = \sigma^{-1}(\varphi(a))\varphi(b)$. It extends to an isomorphism φ_e from $[A]_{\sigma_1}$ onto $[D]_\sigma$ with $\varphi_e \circ \sigma_1 = \sigma \circ \varphi_e$ (see Proposition 4.4).

A closed ideal J of A is (A, σ_1)-semi-invariant (respectively has (A, σ_1)-cancellation) if and only if $h(J)A \subset J$ (respectively $h(A) \cap (A + \mathcal{M}(A, J)) \subset J$) by Remark 4.19.

Remark 4.25. By (HR), it holds

$$\sigma^{-n}(D) + \cdots + \sigma^{-1}(D) + D = h_{1,\infty}(h^n(A) + \cdots + h(A) + A)\,.$$

Hence, $D_{-\infty,1} = h_{1,\infty}(C)$ for the closure C of $A + h(A) + h^2(A) + \cdots$.

$h \colon C \to C$ is a non-degenerate faithful endomorphism, and $[D]_\sigma$ is isomorphic to the inductive limit of $C \xrightarrow{h} C \xrightarrow{h} \cdots$, because $h_{1,\infty} \circ h = \sigma^{-1} \circ h_{1,\infty}$ and $[D]_\sigma$ is in a natural way the inductive limit of $D_{-\infty,1} \xrightarrow{\sigma^{-1}} D_{-\infty,1} \xrightarrow{\sigma^{-1}} \cdots$ (cf. Remark 4.12).

C is of type I if D is of type I.

Corollary 4.26. *Suppose that $h \colon A \to \mathcal{M}(A)$ is a non-degenerate *-monomorphism with $h(A) \cap A = \{0\}$. Let B, $h_{1,\infty} \colon \mathcal{M}(A) \to B$, and $\sigma \in \mathrm{Aut}(B)$ as above. Then:*

(i) $D := h_{1,\infty}(A)$ is σ-modular in the sense of Definition 4.2.

(ii) The Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H}(A, h))$ is the full hereditary C-subalgebra of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ generated by D.*

*(iii) Every non-zero closed ideal I of $\mathcal{O}(\mathcal{H}(A, h))$ has non-zero intersection $D \cap I$ with D. (I.e. every *-representation $\varrho \colon \mathcal{O}(\mathcal{H}(A, h)) \to \mathcal{L}(H)$ with faithful restriction $\varrho|D$ is itself faithful.)*

(iv) $J := (h_{1,\infty})^{-1}(D \cap I)$ satisfies $h(J)A \subset J$.

Proof. (i) is Lemma 4.23(i).

(ii) is part of Proposition 3.6.

Ad (iii): Since $\mathcal{O}(\mathcal{H}(A,h))$ is a full hereditary C*-subalgebra of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$, every non-zero closed ideal is the intersection of a non-zero closed ideal K of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ with $\mathcal{O}(\mathcal{H}(A,h))$. By Proposition 4.5 the intersection $I \cap D = K \cap D$ is non-zero.

Ad (iv): Since $I \cap D = K \cap D$ for a closed ideal K of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$, $I \cap D$ is (D,σ)-semi-invariant in the sense of Definition 4.14 by Proposition 4.5. Thus, $J := h_{1,\infty}^{-1}(I \cap D)$ satisfies $h(J)A \subset J$ by Lemma 4.23(ii). □

Remark 4.27. Conversely, if B_1 and $\sigma_1 \in \mathrm{Aut}\,(B_1)$ and σ_1-modular $D_1 \subset B_1$ are given, then $h_1 \colon D_1 \to \mathcal{M}\,(D_1)$ defined by $h_1(a)b := \sigma_1^{-1}(a)b$ is a non-degenerate *-monomorphism with $h_1(D_1) \cap D_1 = \{0\}$. Then $h_{1,\infty} \colon D_1 \to D$ extends to an isomorphism from $([D_1]_{\sigma_1}, \sigma_1)$ onto $([D]_\sigma, \sigma)$ where $h_{1,\infty}$ and $\sigma \in \mathrm{Aut}\,(B)$ are as in Corollary 4.26. (See Remark 4.19, Corollary 4.20 and 4.4.)

Corollary 4.28. *If $h \colon A \to \mathcal{M}\,(A)$ is a non-degenerate *-monomorphism with $h(A) \cap A = \{0\}$, and A is stable and σ-unital, then the Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H}(A,h))$ is stable and is isomorphic to the crossed product of the inductive limit $C \xrightarrow{h} C \xrightarrow{h} \cdots$ by $\mathbb{Z}$, where C is the closure of $A + h(A) + h^2(A) + \cdots$.*

C has a decomposition series with intermediate factors isomorphic to A. In particular C is of type I if A is of type I.

Proof. Since $\mathcal{O}(\mathcal{H}(A,h))$ is the full hereditary C*-subalgebra of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ generated by D (cf. Corollary 4.26) and since D is isomorphic to A, we have that $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ is stable and is isomorphic to $\mathcal{O}(\mathcal{H}(A,h))$ by Proposition 3.6. On the other hand $[D]_\sigma$ is isomorphic to the inductive limit $C \xrightarrow{h} C \xrightarrow{h} \cdots$ by Remark 4.25. C has a decomposition series given by the closed ideals $C_n := A + h(A) + \cdots + h^n(A)$ with $C_{n+1}/C_n \cong h^{n+1}(A) \cong A$. □

Remark 4.29. More generally, if $\mathcal{P}$ is a property of C*-algebras that is preserved under split-extensions, Morita equivalence, crossed products by $\mathbb{Z}$, and inductive limits, and if A has property $\mathcal{P}$, then $\mathcal{O}(\mathcal{H}(A,h))$ has property $\mathcal{P}$. This happens for example for the property $\mathcal{P}$ of being locally-reflexive, exact, nuclear or weakly-injective.

Corollary 4.30. *Suppose that $h\colon A \to \mathcal{M}(A)$ is a non-degenerate *-monomorphism with $h(A) \cap A = \{0\}$. Build B, $\sigma \in \mathrm{Aut}(B)$ and $D \subset B$ as above from h.*

If I is a closed ideal of $\mathcal{O}(\mathcal{H}(A,h))$ such that $J := (h_{1,\infty})^{-1}(D \cap I)$ satisfies $h(J) = h(A) \cap (A + \mathcal{M}(A,J))$, then I is the closed linear span of $\mathcal{O}(\mathcal{H}(A,h))(D \cap I)\mathcal{O}(\mathcal{H}(A,h))$.

In particular, every ideal K of $\mathcal{O}(\mathcal{H}(A,h))$ with $D \cap I = D \cap K$ coincides with I.

Proof. By Corollary 4.26(iv) and Lemma 4.23(ii) $J_1 := D \cap I$ is (D,σ)-semi-invariant in the sense of Definition 4.2. It has (D,σ)-cancellation by Lemma 4.23(iii). Thus, there is only one closed ideal I_1 of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ with $I_1 \cap D = h_{1,\infty}(J) = D \cap I$. Since $\mathcal{O}(\mathcal{H}(A,h))$ is a full hereditary C*-subalgebra of $[D]_\sigma \rtimes_\sigma \mathbb{Z}$ by Corollary 4.26(ii), it follows that the closed ideal of $\mathcal{O}(\mathcal{H}(A,h))$ generated by $D \cap I$ coincides with I. ☐

Corollary 4.31. *Suppose that $h\colon A \to \mathcal{M}(A)$ is a non-degenerate *-monomorphism with $h(A) \cap A = \{0\}$. Build B, $\sigma \in \mathrm{Aut}(B)$ and $D \subset B$ as above from h.*

If every closed ideal J of A with $h(J)A \subset J$ satisfies

$$h(J) = h(A) \cap (A + \mathcal{M}(A,J)),$$

then the map

$$I \in \mathbb{I}(\mathcal{O}(\mathcal{H}(A,h))) \mapsto I \cap D \in \mathbb{I}(D),$$

is a lattice isomorphism from $\mathbb{I}(\mathcal{O}(\mathcal{H}(A,h))) \cong \mathbb{O}(\mathrm{Prim}(\mathcal{O}(\mathcal{H}(A,h))))$ onto the Hausdorff lattice of closed ideals J of A that satisfy $h(J)A \subset J$.

In particular, $D \cong A$ is a regular C-subalgebra of $\mathcal{O}(\mathcal{H}(A,h))$.*

Corollary 4.31 derives from Corollary 4.30 as Corollary 4.18 derives from Proposition 4.16.

In particular, we get the following corollary.

Corollary 4.32. *Suppose that $h\colon A \to \mathcal{M}(A)$ is a non-degenerate *-monomorphism with $h(A) \cap A = \{0\}$. If every closed ideal with $h(J)A \subset J$ is trivial, then $\mathcal{O}(\mathcal{H}(A,h))$ is simple.*

5 Proofs of the main results

We give here the proof of Theorem 1.4. Further we derive two corollaries, whose contents we have mentioned already in the introduction.

Proof of Theorem 1.4. Recall that $\mathbb{O}(X)$ denotes the open subsets of X. By (I)–(IV), $\Psi(\mathbb{O}(X))$ is a sublattice of $\mathbb{O}(P)$ that contains $\emptyset$, P and is closed under l.u.b. and g.l.b. Let $A := C_0(P, \mathbb{K})$ and $h\colon A \to \mathcal{M}(A)$ the non-degenerate *-monomorphism of Corollary 2.18 for $\Omega := \Psi(\mathbb{O}(X))$. Since h is unitarily equivalent to $\delta_\infty \circ h$ by (ii) of Corollary 2.18 and since $\delta_\infty(\mathcal{M}(A)) \cap A = \{0\}$ by Remark 2.4(iii), it follows that $h(A) \cap A = \{0\}$.

By Remark 2.20 it holds for every $J \in \mathbb{I}(A)$ with $h(J)A \subset J$ that

$$h(J) = h(A) \cap (A + \mathcal{M}(A, J)).$$

Thus Corollary 4.26 and Corollary 4.31 apply, and we get:

(i) $\mathcal{O}(\mathcal{H}(A, h))$ is isomorphic to the crossed product of $E := [D]_\sigma$ by $\mathbb{Z}$ with respect to the shift automorphism σ restricted to E, where $D := h_{1,\infty}(A)$.

(ii) The map
$$I \in \mathbb{I}(\mathcal{O}(\mathcal{H}(A, h))) \mapsto I \cap D \in \mathbb{I}(D),$$
is a lattice isomorphism from $\mathbb{I}(\mathcal{O}(\mathcal{H}(A, h))) \cong \mathbb{O}(\mathrm{Prim}(\mathcal{O}(\mathcal{H}(A, h))))$ onto the Hausdorff lattice of closed ideals J of A that satisfy $h(J)A \subset J$.

By construction of $h\colon A \to \mathcal{M}(A)$ the Hausdorff lattice of closed ideals J of A with $h(J)A \subset J$ is naturally isomorphic to Ω, and Ω is isomorphic to the lattice $\mathbb{O}(X)$ by Ψ. Thus the composition of the isomorphisms define a lattice isomorphism from $\mathbb{O}(\mathrm{Prim}(\mathcal{O}(\mathcal{H}(A, h))))$ onto $\mathbb{O}(X)$. Since X is point-complete by assumption and since $\mathcal{O}(\mathcal{H}(A, h))$ is separable, we get that X and $\mathrm{Prim}(\mathcal{O}(\mathcal{H}(A, h)))$ are homeomorphic by Corollary A.12.

The properties (i)–(iii) follow from Corollary 4.26, Corollary 4.31, and Corollary 4.28. ☐

Remark 5.1. (i) Since $E = [A]_\sigma$ in our proof and since every σ-invariant ideal I of $[A]_\sigma$ is determined by its intersection with $A = C_0(P, \mathbb{K})$, we have that $C_0(P)$ is isomorphic to a regular Abelian C*-subalgebra of $E \rtimes_\sigma \mathbb{Z}$.

(ii) If B is a separable C*-algebra such that $B \otimes \mathcal{O}_2$ contains an Abelian regular subalgebra then Theorem 1.4 applies to $X := \mathrm{Prim}\,(B \otimes \mathcal{O}_2) \cong \mathrm{Prim}\,(B)$ by Lemma A.14. Further, in the case where B is in addition nuclear, we get that $B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is isomorphic to $(E \rtimes_\sigma \mathbb{Z}) \otimes \mathcal{O}_2 \otimes \mathbb{K}$ from [15, chp. 1, cor. L] (cf. Remark 2.17).

Theorem 1.4 and Remark 5.1 imply:

Corollary 5.2. *If B is a separable* C*-*algebra such that $B \otimes \mathcal{O}_2$ contains an Abelian regular* C*-*subalgebra C, then the primitive ideal space of B is isomorphic to the primitive ideal space of the crossed product $E \rtimes_\sigma \mathbb{Z}$ of an inductive limit E of* C*-*algebras of type I by an automorphism σ of E.*

In particular, if B is in addition nuclear, then

$$B \otimes \mathcal{O}_2 \otimes \mathbb{K} \cong (E \rtimes_\sigma \mathbb{Z}) \otimes \mathcal{O}_2 \otimes \mathbb{K}.$$

Corollary 5.3. *A separable* C*-*algebra B has a primitive ideal space $X =$* $\mathrm{Prim}\,(B)$ *which is the continuous pseudo-open and pseudo-epimorphic image of a locally compact Polish space P if and only if $B \otimes \mathcal{O}_2$ contains a "regular" Abelian* C*-*subalgebra in the sense of Definition 1.2.*

Proof. If $B \otimes \mathcal{O}_2$ contains an Abelian C*-subalgebra $C \cong C_0\,(P)$ which is regular in $B \otimes \mathcal{O}_2$, then $X = \mathrm{Prim}\,(B) \cong \mathrm{Prim}\,(B \otimes \mathcal{O}_2)$ is the pseudo-open and pseudo-epimorphic image of P by Corollary 1.5.

Conversely if $\pi\colon P \to X$ is pseudo-open and pseudo-epimorphic, then there is a *-monomorphism φ from $C_0\,(P)$ into $B \otimes \mathcal{O}_2$ which is given by the embeddings

$$C_0\,(P) \cong C_0\,(P, \mathbb{C} \cdot 1 \otimes e_{11}) \subset C_0\,(P, \mathcal{O}_2 \otimes \mathbb{K}) \hookrightarrow B \otimes \mathcal{O}_2 \otimes \mathbb{K} \subset B \otimes \mathcal{O}_2$$

such that $\Psi(U) := \pi^{-1}(U)$ is induced by $J \mapsto \varphi^{-1}(\varphi(C_0\,(P)) \cap J)$. Here the inclusion $C_0\,(P, \mathcal{O}_2 \otimes \mathbb{K}) \hookrightarrow B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ comes from [15, chp. 1, thm. K] because $B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is stable and strongly purely infinite. □

Remark 5.4. The construction used in the Theorem 1.4 implies, for $X = \{\text{point}\}$, that $E \rtimes_\sigma \mathbb{Z}$ is isomorphic to $\mathcal{O}_\infty \otimes \mathbb{K} = \mathcal{O}\,(\ell_2) \otimes \mathbb{K}$. If $X = P$ is Hausdorff and $\Psi = \mathrm{id}_P$, then $E \rtimes_\sigma \mathbb{Z}$ is isomorphic to $C_0\,(P, \mathcal{O}_\infty \otimes \mathbb{K})$.

6 Dini spaces

Throughout this section we require that X is a point-complete and second countable T_0-space. (If one of this conditions is not satisfied, then Definitions and results become more complicate.)

Definition 6.1. A function $g\colon X \to [0,\infty)$ is a *Dini function* on X if g is a lower semi-continuous and $\sup g\left(\bigcap_n F_n\right) = \inf_n \sup g\left(F_n\right)$ for every decreasing sequence $F_1 \supset F_2 \supset \dots$ of closed subsets F_n of X. (Here we use the convention $\sup \emptyset := 0$.)

We call X a *Dini space* if the supports of Dini functions build a base of the topology of X.

It turns out that

(i) If g, h are Dini functions on X then g is bounded and $\max(g,h)$ is a Dini functions.

(ii) The set of Dini functions on X is closed under uniform convergence.

(iii) If g is a Dini function and $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing continuous function with $f(0) = 0$ then $x \mapsto f(g(x))$ is a Dini function.

In [16] the second-named author has shown that for a non-negative lower semi-continuous function g on X the following three properties (iv)-(vi) are equivalent (and justify the name "Dini" function):

(iv) g is a Dini function.

(v) Every increasing sequence $0 \leq f_1 \leq f_2 \leq \dots$ of non-negative lower semi-continuous functions on X with $g(x) = \sup_n f_n(x)$ for all $x \in X$ converges uniformly to g (i.e. $\lim_n \|g - f_n\|_\infty = 0$)

(vi) For every $\gamma > 0$ the G_δ-set $\{y \in X : \; g(y) \geq \gamma\}$ is quasi-compact and g is bounded.

In general the set of Dini functions is not convex and is not closed under multiplication or minima, because of the following result in [17]:
For separable C*-algebra A the Dini functions g on $\mathrm{Prim}\,(A)$ are nothing else than the generalized Gelfand transforms $g = N(a)$ of elements $a \in A$ given by $N(a)(J) = \|a + J\|$ for primitive ideals J of A.

It has been shown in [16] that a X point-complete second-countable T_0 space is a Dini space if and only if X locally quasi-compact. In particular, primitive ideal spaces of separable C*-algebras are Dini spaces.

All Dini spaces are continuous and open images of Polish spaces [18]. Continuous and open images X of Polish spaces have l.u.b. and g.l.b. compatible isomorphic embeddings of their lattices of open subsets into lattices of open subsets of Polish spaces. Unfortunately a Polish space P is the pseudo-open and pseudo-epimorphic image of a locally compact space Q if and only if P is itself locally compact. Thus there is still an *open question* whether every Dini space is isomorphic to the primitive ideal space of a separable C*-algebra or not.

Our results yield equivalent descriptions of primitive ideal spaces of separable nuclear C*-spaces among the Dini spaces, in fact, the following properties (a)–(e) of a Dini space X are equivalent:

(a) X is isomorphic to the primitive ideal space of a separable *nuclear* C*-algebra.

(b) $\mathbb{F}(X)$ is lattice-isomorphic to a sub-lattice $\mathcal{G}$ of $\mathbb{F}(Y)$ which is closed under forming of l.u.b. and g.l.b. for some *locally compact* Polish space Y. (Here $\mathbb{F}(Y)$ means the lattice of closed subsets of Y. The g.l.b. is just the intersection, and the l.u.b. is the closure of the union of a family in $\mathbb{F}(Y)$.) I.e., there is a map Ψ from the open subsets $\mathbb{O}(X)$ of X into the open subsets $\mathbb{O}(Y)$ of Y with properties (I)–(IV) of Definition 1.1.

(c) $\mathbb{F}((0,1]_{lsc} \times X)$ is the projective limit of $\mathbb{F}(P_n \setminus \{q_n\})$ for pointed finite one-dimensional polyhedral (P_n, q_n) (in a lattice sense). The connecting maps $\Phi_n \colon \mathbb{F}(P_{n+1} \setminus \{q_{n+1}\}) \to \mathbb{F}(P_n \setminus \{q_n\})$ satisfy (with $Y_n = P_n \setminus \{q_n\}$):

  (I$_0$') $\Phi_n(Y_{n+1}) = Y_n\,,\ \ \Phi_n(\emptyset) = \emptyset$.

  (II') $\Phi_n(\overline{\bigcup_\tau F_\tau}) = \overline{\bigcup_\tau \Phi_n(F_\tau)}$ for every family $\{F_\tau\}_\tau$ of closed subsets of $\mathbb{F}(Y_{n+1})$,

  (III$_0$') $\Phi_n(\bigcap_k F_k) = \bigcap_k \Phi_n(F_k)$ for every *decreasing* sequence $F_1 \supset F_2 \supset \cdots$ in $\mathbb{F}(Y_{n+1})$, and

(d) There are a locally compact Polish space Y and a continuous map $\pi \colon Y \to X$ such that, for closed subset $F \subset G$ of X with $F \neq G$, the set $G \setminus F$ contains a point of $\pi(Y)$, and that

$$\overline{\bigcup_n \pi^{-1}(F_n)} = \pi^{-1}\left(\overline{\bigcup_n F_n}\right)$$

for every increasing sequence $F_1 \subset F_2 \subset \cdots$ of closed subsets of X.

(e) $\mathbb{O}(X)$ is the projective limit of a sequence of maps

$$\Psi_n \colon \mathbb{O}(X_{n+1}) \to \mathbb{O}(X_n)$$

with properties (I), (II) and (III_0) and $X_n \cong \mathrm{Prim}(A_n)$ for a separable exact C*-algebra.

Proof. Ad (b)$\Longleftrightarrow$(d): by Proposition A.11 (note that we can restrict to increasing sequences of closed subsets because X is second countable).

Ad (a)$\Longleftrightarrow$(d): by Corollary 1.5.

Ad (a)$\Longleftrightarrow$(e): Clearly, (a) implies (e) with $X_n = X$ and $\Psi_n = \mathrm{id}_X$.

(e) implies (a) because for every separable exact C*–algebra A there is a *nuclear* stable separable C*-algebra B with the same primitive ideal space and $B \cong B \otimes \mathcal{O}_2$ (cf. [15, cor. 12.2.20]). Then Ψ_n is induced by a non-degenerate *-monomorphism $h_n \colon B_n \hookrightarrow B_{n+1}$ (cf. [15, thm. K]) and $B := \mathrm{indlim}\,(h_n \colon B_n \to B_{n+1})$ has primitive ideal space $\cong X$.

Ad (a)$\Longrightarrow$(c): by [20, Thm. 5.12, Prop. 6.2], because $(0,1]_{lsc} \times X$ is the primitive ideal space of $\mathcal{A}_{[0,1]} \otimes B$ where B is separable and nuclear and $X \cong \mathrm{Prim}(B)$.

Ad (c)$\Longrightarrow$(d): (c) implies that $Z = (0,1]_{lsc} \times X$ satisfies (e) (with Z in place of X). Thus, there is a separable nuclear C*-algebra A with primitive ideal space isomorphic to Z by the implication (e)$\Rightarrow$(a). From the implication (a)$\Rightarrow$(d) it follows that Z is a pseudo-epimorphic and pseudo-open image of a locally compact Polish space Y. The composition $p_2 \circ \pi$ of the map $\pi \colon Y \to Z$ with the projection $p_2 \colon (t,x) \in Z \to x \in X$ is again pseudo-epimorphic and pseudo-open. □

Remark 6.2. For every Dini space X there is a map $\Psi \colon \mathbb{O}(X) \to \mathbb{O}(\mathbb{R}_+ \times \mathbb{N})$ with properties (I), (II), (III_0) and (IV).

If, in addition, X is isomorphic to the primitive ideal space of a separable C*-algebra, then there exists a separable *nuclear* C*-algebra A and an epimorphism $\Phi \colon \mathbb{O}(\mathrm{Prim}(A)) \to \mathbb{O}(X)$ with (I), (II) and (III_0). The latter map Φ is unknown for general Dini spaces.

A Preliminaries

A.1 T_0-spaces

Definition A.1. Suppose that X is a T_0-space (i.e. $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$). X is called
(i) *second countable* if the topology of X contains a countable base,
(ii) *prime* if it is not the union of two closed true subsets of X (and a subset $F \subset X$ is called prime, if it is prime in its relative topology, Hausdorff [9, p. 231] calls a non-prime closed subset *decomposable*), and
(iii) *point-complete* if every closed prime subset of X is the closure of a singleton (the name "spectral space" is used in [11, def. 4.9], others use the terminology "sober space" for our point-complete spaces).

For a topological space X to be prime it is of course equivalent that it does not contain two disjoint open subsets. Equivalently every open subset of X is dense in X. If a subspace F of X is prime, then so is its closure $\overline{F}$, and, by that, the closure of a singleton is prime. In the case where X is the primitive ideal space of a separable C*-algebra A every prime closed set is the closure of a singleton because there is an open and continuous map from the Polish space of pure states on A onto X.

A.2 Maps related to Ψ

Here we give some results on the relation between lattice maps, point maps, and lower semi-continuous selections. We use in this paper Remark A.4, Lemma A.8, Proposition A.11, Corollary A.12, and Lemma A.15.

We adopt a more general viewpoint and suppose that X, Y are T_0 spaces, that $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ satisfies property (III) of Definition 1.1 and the following weaker conditions (I_0) and (II_0) instead of (I) and (II):

(I_0) $\Psi(X) = Y$, $\Psi(\emptyset) = \emptyset$,

(II_0) $\Psi(U \cap V) = \Psi(U) \cap \Psi(V)$ for all open subsets $U, V \subset X$.

For example, if $\pi\colon Y \to X$ is a continuous map, then the map $\Psi: \mathbb{O}(X) \to \mathbb{O}(Y)$ defined by $\Psi(U) := \pi^{-1}(U)$ obviously satisfies properties (I_0), (II_0) and (III). Conversely, for every map $\Psi: \mathbb{O}(X) \to \mathbb{O}(Y)$ with (I_0), (II_0) and (III) there is a unique continuous map $\pi\colon Y \to X$ with $\Psi(U) = \pi^{-1}(U)$ for open $U \subset X$ (cf. Proposition A.11 below).

On account of properties (I_0) and (II_0) one can define a new topology on Y by an interior operation

$$Z^{\circ\Psi} \;:=\; \bigcup\{\Psi(U):\; U \in \mathbb{O}(X)\,,\, \Psi(U) \subset Z\} \subset Z^{\circ}$$

for every subset $Z \subset Y$, or the corresponding closure operation:

$$\overline{Z}^{\Psi} \;:=\; \bigcap\{Y \setminus \Psi(U):\; U \in \mathbb{O}(X)\,,\, \Psi(U) \subset Y \setminus Z\} \subset \overline{Z}\,.$$

If Ψ satisfies property (III), then $\Psi(\mathbb{O}(X))$ is not only a base of this Ψ-topology but is the set of all Ψ-open sets. The Ψ-topology is coarser than the given T_0-topology of Y and is in general not T_0.

Definition A.2. Suppose that X, Y are T_0 and that $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ is an arbitrary map. We call the (order preserving) map

$$\Phi\colon V \in \mathbb{O}(Y) \mapsto \bigcup\{U \in \mathbb{O}(X)\,:\; \Psi(U) \subset V\} \in \mathbb{O}(X)$$

the *pseudo-left-inverse* of Ψ. (It is a left-inverse Φ of Ψ if Ψ satisfies the properties (I), (III) and (IV) of Definition 1.1.)

Lemma A.3. *Suppose that $\Psi : \mathbb{O}(X) \to \mathbb{O}(Y)$ satisfies properties (I_0) and (III) of Definition 1.1. Let Φ denote the pseudo-left-inverse of Ψ, then*

- *(a)* $\Psi(\Phi(V)) \subset V$ *for* $V \in \mathbb{O}(Y)$, $U \subset \Phi(\Psi(U))$ *for* $U \in \mathbb{O}(X)$.
- *(b)* $\Phi^{-1}(X) = \{Y\}$ *and* $\Phi(\emptyset)$ *is the biggest open subset* U_0 *of* X *with* $\Psi(U_0) = \emptyset$,
- *(c)* $\Phi\left(\left(\bigcap_\alpha V_\alpha\right)^{\circ}\right) = \left(\bigcap_\alpha \Phi(V_\alpha)\right)^{\circ}$ *for every family* $\{V_\alpha\}_\alpha \subset \mathbb{O}(Y)$.
- *(d)* $\Phi \circ \Psi \circ \Phi = \Phi$ *and* $\Psi \circ \Phi \circ \Psi = \Psi$.
- *(e)* $\Psi(\Phi(V)) = V^{\circ\Psi}$ *and* $\Phi(V) = \Phi\left(V^{\circ\Psi}\right)$ *for any* $V \in \mathbb{O}(Y)$ *if* Ψ *has in addition property (II_0).*

$\Phi \circ \Psi = \mathrm{id}_{\mathbb{O}(X)}$ *if* Ψ *is injective.*

Clearly (e) means $\Phi\left(Y \setminus \overline{Z}\right) = \Phi\left(Y \setminus \overline{Z}^{\Psi}\right)$ for any subset $Z \subset Y$.

Proof. *Ad (a):* $U \subset \Phi(\Psi(U))$ by Definition A.2.

$\Psi(\Phi(V)) = \bigcup\{\Psi(U): \ U \in \mathbb{O}(X), \Psi(U) \subset V\} \subset V$ by (III) and A.2.

Ad (b): If $\Phi(V) = X$, then $Y = \Psi(X) \subset V$. $\Psi(U) = \emptyset$ implies $U \subset \Phi(\emptyset)$.

Ad (c): By monotony of Φ it follows for $W := (\bigcap_\alpha V_\alpha)^\circ$ and $U := (\bigcap_\alpha \Phi(V_\alpha))^\circ$ that $\Phi(W) \subset \Phi(V_\alpha)$ and $\Phi(W) \subset U$, because $\Phi(W)$ is open. Clearly, $U \subset \Phi(V_\alpha)$ for all α and $U \in \mathbb{O}(X)$.

Then $\Psi(U) \subset \Psi(\Phi(V_\alpha)) \subset V_\alpha$ for all α, because Ψ is increasing by property (III). Since $\Psi(U)$ is open, this implies $\Psi(U) \subset W$. Thus $U \subset \Phi(W)$ by definition of Φ. Hence, $\Phi(W) = U$.

Ad (d): Part (a) implies $\Phi(V) \subset \Phi(\Psi(\Phi(V))) \subset \Phi(V)$, for $U = \Phi(V)$, because Φ is increasing. In the same way: $\Psi(U) \subset \Psi(\Phi(\Psi(U))) \subset \Psi(V)$ with $V := \Psi(U)$, because Ψ is increasing by property (III).

Ad (e): $\Psi(\Phi(V)) = V^{\circ\Psi}$ by property (III). Thus $\Phi(V) = \Phi\left(V^{\circ\Psi}\right)$ by (d).

If Ψ is injective, then $\Psi(U) = \Psi(\Phi \circ \Psi(U))$ implies $U = \Phi \circ \Psi(U)$, i.e. $\Phi \circ \Psi = \mathrm{id}$ by (d). $\square$

Remark A.4. Suppose that $\mathcal{Z}$ is a sub-lattice of $\mathbb{O}(Y)$ that contains Y, $\emptyset$ and l.u.b. and g.l.b. of families $\{U_\alpha\}$ in $\mathcal{Z}$. Then Lemma A.3 implies the existence of an order-preserving map $\Theta\colon \mathbb{O}(Y) \to \mathbb{O}(Y)$ with $\Theta|\mathcal{Z} = \mathrm{id}_\mathcal{Z}$, $\Theta(\mathbb{O}(Y)) = \mathcal{Z}$, $\Theta \circ \Theta = \Theta$, $\Theta(V) \subset V$ for $V \in \mathbb{O}(Y)$ and $\Theta((\bigcap_\alpha V_\alpha)^\circ) = (\bigcap_\alpha \Theta(V_\alpha))^\circ$ for every family $\{V_\alpha\}_\alpha \subset \mathbb{O}(Y)$.

Indeed, $\mathcal{Z}$ is the set of all open sets of a coarser topology on Y. Let R denote the equivalence relation $x \sim_\mathcal{Z} y$ given by $\overline{\{x\}}^\mathcal{Z} = \overline{\{y\}}^\mathcal{Z}$. The quotient space $X := Y/R$ is a T_0-space and the map $\pi\colon y \in Y \to [y]_R \in Y/R$ to the equivalence classes is continuous. Moreover $\Psi(U) := \pi^{-1}U$ defines a lattice isomorphism Ψ from $\mathbb{O}(X)$ onto $\mathcal{Z} \subset \mathbb{O}(Y)$. $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ satisfies (I)–(IV) of Definition 1.1, because $\mathcal{Z}$ is closed under unions and $\mathbb{O}(Y)$-interiors of intersections of families in $\mathcal{Z}$. Let $\Phi\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ the left-inverse defined by Lemma A.3 and let $\Theta := \Phi \circ \Psi$. Then $\Theta\colon \mathbb{O}(Y) \to \mathbb{O}(Y)$ has the above listed properties by Lemma A.3.

Lemma A.5. *Suppose that X and Y are T_0 spaces and that $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ satisfies properties (III) of Definition 1.1,*

(I_0) $\Psi(X) = Y$, $\Psi(\emptyset) = \emptyset$, *and*

(II_0) $\Psi(U \cap V) = \Psi(U) \cap \Psi(V)$ *for all open subsets* $U, V \subset X$.

Let Φ denote the pseudo-left-inverse of Ψ. Then for any prime closed set $F \subset Y$ the set $X \setminus \Phi(Y \setminus F)$ is a prime closed subset of X.

In particular, if X is point-complete, then the complement F_y of

$$\Phi\left(Y \setminus \overline{\{y\}}\right) = \bigcup \{U \in \mathbb{O}(X) : \ y \notin \Psi(U)\}$$

is the closure of a singleton $x =: \pi(y)$ for every $y \in Y$.

Proof. Suppose $X \setminus \Phi(Y \setminus F)$ is not prime, i.e. $\Phi(Y \setminus F)$ is the intersection of two open subsets $U_1, U_2 \subset X$ both different from $\Phi(Y \setminus F)$. Then $\Psi(\Phi(Y \setminus F)) \subset \Psi(U_j)$ for $j = 1, 2$ and $\Psi(\Phi(Y \setminus F)) = \Psi(U_1 \cap U_2) = \Psi(U_1) \cap \Psi(U_2)$ by property (II_0) and monotony of Ψ. Further $\Psi(U_1) \cap \Psi(U_2) = \Psi(\Phi(Y \setminus F)) \subset Y \setminus F$. This is equivalent to $F \subset Y \setminus (\Psi(U_1 \cap \Psi(U_2))$ and implies $F = (F \cap (Y \setminus \Psi(U_1))) \cup (F \cap (Y \setminus \Psi(U_2)))$, i.e. F is the union of two closed sets. But both sets $F \cap (Y \setminus \Psi(U_j))$, $j = 1, 2$ are different from F, since $U_j \not\subset \Phi(Y \setminus F)$ implies $\Psi(U_j) \not\subset Y \setminus F$ for $j = 1, 2$. So F is not prime. $\square$

By Lemma A.5 we get a well-defined map

$$\Phi' \colon F \in Y^c := \mathrm{prime}\,(\mathbb{F}(Y)) \mapsto X \setminus \Phi(Y \setminus F) \in X^c := \mathrm{prime}\,(\mathbb{F}(X))$$

One can define maps η that map every point to its closure, i.e.

$$\begin{aligned} \eta_Y \colon \ y \in Y \quad &\mapsto \quad \overline{\{y\}} \in \mathrm{prime}\,(\mathbb{F}(Y)) \\ \eta_X \colon \ x \in X \quad &\mapsto \quad \overline{\{x\}} \in \mathrm{prime}\,(\mathbb{F}(X)), \end{aligned}$$

which are injective because X, Y are T_0-spaces. If $\eta_X(X) = \mathrm{prime}\,(\mathbb{F}(X)))$, i.e. if X is point-complete, then the map $\pi \colon Y \to X$ of Lemma A.5 is given by $\pi := (\eta_X)^{-1} \circ \Phi' \circ \eta_Y$ and satisfies $\overline{\{\pi(y)\}} = \Phi'(\overline{\{y\}})$ for $y \in Y$.

Lemma A.6. *Suppose that $\Psi \colon \mathbb{O}(X) \to \mathbb{O}(Y)$ satisfies the assumptions of Lemma A.5 and that X is point-complete. Let $\pi \colon Y \to X$ as above. Then*

$$\pi^{-1}U = \Psi(U) \qquad \textit{for} \quad U \in \mathbb{O}(X).$$

In particular, $\pi \colon Y \to X$ is continuous.

Proof. $\pi^{-1}(X \setminus F) = Y \setminus \pi^{-1}(F)$ for any subset $F \subset X$, $\pi^{-1}(X) = Y$, and $\pi^{-1}(\emptyset) = \emptyset$ hold for any map $\pi \colon Y \to X$.

Consider a prime set $F \in \operatorname{prime}(\mathbb{F}(Y))$. Then, for $U \in \mathbb{O}(X)$,

$$\begin{aligned}
&\Phi'(F) = X \setminus \Phi(Y \setminus F) \subset X \setminus U \\
&\iff U \subset \Phi(Y \setminus F) \quad \iff \quad \Psi(U) \subset Y \setminus F \\
&\iff F \subset Y \setminus \Psi(U) \in \mathbb{F}(Y) \quad \iff \quad \forall y \in F \colon y \in Y \setminus \Psi(U),
\end{aligned}$$

where the second equivalence comes from Definition A.2 of Φ and property (III) of Ψ. In particular,

$$\overline{\{\pi(y)\}} = \Phi'\left(\overline{\{y\}}\right) \subset X \setminus U \quad \iff \quad y \in Y \setminus \Psi(U),$$

and this shows that $\pi(y) \in X \setminus U \in \mathbb{F}(X)$ if and only if $y \in Y \setminus \Psi(U)$. Thus $\pi^{-1}U = \Psi(U)$. □

Let X and Y T_0-spaces, we let $p_1 \colon Y \times X \to Y$ and $p_2 \colon Y \times X \to X$ the maps $p_1(y, x) := y$ and $p_2(y, x) := x$ to the components.

Lemma A.7. *Suppose that $R \subset Y \times X$ satisfies $p_1(R) = Y$. Let $\lambda(y)$ denote the subset $p_2(p_1^{-1}(y) \cap R)$ of X for $y \in Y$.*

(i) $p_1 \colon R \to Y$ is open as a map from R to Y, if and only if,

$$\lambda(y) \subset \overline{\bigcup_{v \in V} \lambda(v)}$$

for every subset $V \subset Y$ and $y \in \overline{V}$.

(ii) If X is a compact convex set, ∂X the set of its extreme points and if $\lambda(y)$ is the closed convex span of $\lambda(y) \cap \partial X$ for every $y \in Y$, then $p_1 \colon R \to Y$ is open if and only if $p_1 \colon R \cap (Y \times \partial X) \to Y$ is open.

Proof. Ad (i): Let α denote the restriction of p_1 to R. Note that $\bigcup_{v \in V} \lambda(v)$ is equal to $p_2\left(\alpha^{-1}(V)\right) = p_2\left(p_1^{-1}(V) \cap R\right)$ for every subset V of Y, because $\alpha^{-1}(V)$ is the union of the sets $\{v\} \times \lambda(v)$ with $v \in V$.

It is easy to see that the continuous map $\alpha \colon R \to Y$ is open if and only if $(\alpha^{-1}(W))^\circ \subset \alpha^{-1}(W^\circ)$ for every subset W of Y (cf. eg. [16, lem. 6.9]). And this is equivalent to $\alpha^{-1}\left(\overline{V}\right) \subset \overline{\alpha^{-1}(V)}$ for every subset V of Y (by $W := X \setminus V$). On the other hand, $p_2\left(\overline{S}\right) \subset \overline{p_2(S)}$ for every subset S of R, because $p_2 \colon R \to X$ is continuous. Thus $\lambda(y)$ is contained in the closure of $p_2\left(\alpha^{-1}(V)\right)$ for every subset $V \subset Y$ and every $y \in \overline{V}$ if $\alpha \colon R \to Y$ is open.

Conversely, suppose that $p_2\left(\alpha^{-1}(\overline{V})\right)$ is contained in $\overline{p_2\left(\alpha^{-1}(V)\right)}$ for every subset $V \subset Y$.

The complement of $\overline{\alpha^{-1}(V)}$ is the union of Cartesian products $U_1 \times U_2$ for $U_1 \subset Y$ and $U_2 \subset X$ open with $(U_1 \times U_2) \cap \alpha^{-1}(V) = \emptyset$.

Then $\alpha^{-1}(V \cap U_1) \subset Y \times (X \setminus U_2)$, thus $p_2\left(\alpha^{-1}(V \cap U_1)\right) \subset X \setminus U_2$. By assumptions, $\lambda(y) \subset X \setminus U_2$ for $y \in \overline{V} \cap U_1$ because $X \setminus U_2$ is closed and $\overline{V} \cap U_1 \subset \overline{V \cap U_1}$. It implies $p_2(\alpha^{-1}(\overline{V}) \cap (U_1 \times X)) \subset X \setminus U_2$ and, finally, $\alpha^{-1}(\overline{V}) \cap (U_1 \times U_2) = \emptyset$. It shows $\alpha^{-1}(\overline{V}) \subset \overline{\alpha^{-1}(V)}$ for every subset of V of Y, i.e. α is open.

Ad (ii): Recall that a compact convex set C is the closed convex hull of the set ∂C of the extreme points of C (Krein-Milman theorem), and that ∂C is contained in the closure $\overline{S}$ of every subset $S \subset C$ such that C is the closed convex hull of S. Further, $p_2\left(p_1^{-1}(y) \cap R \cap (Y \times \partial X)\right)$ is the same as $\lambda(y) \cap \partial X$ for $y \in Y$.

$\bigcup_{v \in V} \lambda(v)$ is contained in the closed convex hull C of $\bigcup_{v \in V}(\lambda(v) \cap \partial X)$, because $\lambda(v)$ is the closed convex hull of $\lambda(v) \cap \partial X$.

Thus $\lambda(y) \cap \partial X \subset \partial C$ is contained in the closure of $\bigcup_{v \in V}(\lambda(v) \cap \partial X)$, if $\lambda(y)$ is contained in the closure of $\bigcup_{v \in V} \lambda(v)$. Hence, $p_1 \colon R \cap (Y \times \partial X) \to Y$ is open if $p_1 \colon R \to Y$ is open by part(i).

We need a sort of "uniform" lower semi-continuity of $y \mapsto \lambda(y) \cap \partial X$ for the proof of the other direction: Suppose that $p_1 \colon R \cap (Y \times \partial X) \to Y$ is open, $V \subset Y$ and $y \in \overline{V}$, $x_1, \ldots, x_n \in \lambda(y) \cap \partial X$ and that $U_1, \ldots, U_n$ are open subsets of X with $x_j \in U_j$ for $j = 1, \ldots, n$. We show that the closure of the set

$$V_k := \{v \in V : \ U_j \cap \lambda(v) \cap \partial X \neq \emptyset \quad for \quad j = 1, \ldots, k\}$$

contains y for $k \leq n$. We proceed by induction over $k = 1, \ldots, n$. (The case $k = 1$ goes with $V_0 := V$ as the induction step.)

Let $U_0 \subset Y$ denote an open neighborhood of y. Since $\overline{V_k} \cap U_0 \subset \overline{V_k \cap U_0}$, y is in the closure of $V_k \cap U_0$, thus, $x_{k+1} \in \lambda(y) \cap \partial X$ is contained in the closure of $\bigcup_{v \in V_k \cap U_0}(\lambda(v) \cap \partial X)$. Hence, $V_{k+1} \cap U_0 \neq \emptyset$. It shows that y is in the closure of V_{k+1}.

Now, if $y \in \overline{V}$ and $x \in \lambda(y)$, W a neighborhood of x, then there are $x_1, \ldots, x_n \in \lambda(y) \cap \partial X$, $\mu_1, \ldots, \mu_n \in [0,1]$ with $\sum_k \mu_k = 1$, and open neighborhoods U_k of x_k in X such that $\sum_k \mu_k z_k \in W$ whenever $z_k \in U_k$ for $k = 1, \ldots, n$, because $\lambda(y)$ is the closed convex span of $\lambda(y) \cap \partial X$ and $(z_1, \ldots, z_n) \in X^n \to \sum_k \mu_k z_k \in X$ is continuous.

There are $v \in V$ and $z_j \in U_j \cap \lambda(v) \cap \partial X$ by the "uniform" lower semi-continuity of $y \mapsto \lambda(y) \cap \partial X$. $\sum_k \mu_k z_k$ is in $\lambda(v) \cap W$. Thus $\lambda(y)$ is contained in the closure of $\bigcup_{v \in V} \lambda(v)$, if $y \in \overline{V}$, and $p_1 \colon R \to X$ is open by part (i). $\square$

Lemma A.8. *Let $\pi : Y \to X$ be a continuous and surjective map. Then the following are equivalent:*

(i) $\overline{\bigcup_\alpha \pi^{-1} F_\alpha} = \pi^{-1}\left(\overline{\bigcup_\alpha F_\alpha}\right)$ *holds for every increasing family $\{F_\alpha\}_{\alpha \in I}$ of closed subsets of X.*

(ii) $\left(\bigcap_\alpha \pi^{-1} U_\alpha\right)^\circ = \pi^{-1}\left(\left(\bigcap_\alpha U_\alpha\right)^\circ\right)$ *holds for every decreasing family $\{U_\alpha\}$ of open subsets of X.*

(iii) Every R_π-invariant open subset of Y is in $\pi^{-1}\mathbb{O}(X)$, and the map $(y, z) \in R_\pi \mapsto y \in Y$ is an open and surjective map.

(iv) For every closed subset G of Y, the set

$$F_G := \left\{ x \in X : \ \pi^{-1}\left(\overline{\{x\}}\right) \subset G \right\}$$

is closed in X.

Here $R_\pi \subset Y \times Y$ denotes the partial order on Y defined by π. Recall that

$$R_\pi := \left\{ (y, z) \in Y \times Y : \ \overline{\{\pi(y)\}} \ni \pi(z) \right\}$$

and that $V \subset Y$ is R_π-*invariant* if $z \in V$ and $(y, z) \in R_\pi$ implies $y \in V$, cf. Definition 1.3. It means that $y \in Y \setminus V$ if and only if $\pi^{-1}\left(\overline{\{y\}}\right) \subset Y \setminus V$.

Note that $\pi^{-1}\left(\overline{\{y\}}\right) = p_2(p_1^{-1}(y) \cap R_\pi)$ where $p_i \colon Y \times Y \to Y$ is the projection onto the i-th component.

Proof. The equivalence of (i) and (ii) can be seen easily by passing to the complements.

We establish another equivalence for later use: (i) holds, if and only if,

$$\overline{\bigcup_{z \in Z} \pi^{-1}(\overline{\{z\}})} = \pi^{-1}\left(\overline{Z}\right) \qquad \text{for all} \quad Z \subset X\,. \tag{A.1}$$

Indeed: (A.1) is a special case of (i) with $F_z := \overline{\{z\}}$ for $z \in Z$, because $\overline{Z} = \overline{\bigcup_{z \in Z} \overline{\{z\}}}$. Conversely, (A.1) implies (i) because $\pi^{-1}(F) = \bigcup_{z \in F} \pi^{-1}\left(\overline{\{z\}}\right)$ for closed $F \subset X$.

Ad (i)⇒(iii): We show that (A.1) implies (iii). Let V a subset of Y, $Z := \pi(V)$ in (A.1), and let $\lambda(y) := p_2\left(p_1^{-1}(y) \cap R_\pi\right) = \pi^{-1}(\overline{\pi(y)})$ for $y \in Y$. (A.1) implies that $\lambda(y)$ is in the closure of $\bigcup_{v \in V} \lambda(v) = \bigcup_{z \in Z} \overline{\{z\}}$ if y is in the closure of V, because $\overline{\{\pi(y)\}} \subset \overline{Z}$. Thus $p_1 \colon R_\pi \to Y$ is open by Lemma A.7(i).

Suppose that V is an R_π-invariant open subset of Y, and let $F := Y \setminus V$, $Z := \pi(F)$. Then F is closed, and $y \in F$ implies $\pi^{-1}\left(\overline{\{\pi(y)\}}\right) \subset F$. By (A.1), $\pi^{-1}(\overline{Z}) \subset F$. Since $F \subset \pi^{-1}(\pi(F))$, we get $Z = \overline{Z}$ $F = \pi^{-1}(Z)$, and $V = \pi^{-1}(U)$ for the open set $U = X \setminus Z$.

Ad (iii)⇒(iv): Let $\lambda(y) := p_2(p_1^{-1}(y) \cap R_\pi) = \pi^{-1}\left(\overline{\{\pi(y)\}}\right)$ for $y \in Y$. If G is a closed subset of Y let $W := \{y \in Y : \ \lambda(y) \subset G\}$. Then $F_G = \pi(W)$, and W is closed, because $\lambda(y)$ is in the closure of $\bigcup_{w \in W} \lambda(w)$ and G is closed. If $y \in W$ and $z \in \lambda(y)$ then $\pi(z) \in \overline{\{\pi(y)\}}$. Hence $\lambda(z) \subset \lambda(y)$ and $z \in W$. It means that $V := Y \setminus W$ is an R_π-invariant open subset of Y. By (iii), $V = \pi^{-1}(U)$ for an open subset U of X, i.e. $\pi^{-1}(X \setminus U) = W$ and $F_G = X \setminus U$ is closed.

Ad (iv)⇒(i): Let $Z \subset X$ and $G := \overline{\bigcup_{z \in Z} \pi^{-1}(\overline{\{z\}})}$. Then $G \subset \pi^{-1}(\overline{Z})$, and $Z \subset F_G$ and $\pi^{-1}(F_G) \subset G$ by definition of F_G in (iv). Since F_G is closed by (iv), $\overline{Z} \subset F_G$. Thus $\pi^{-1}(\overline{Z}) = G$. I.e. (iv) implies (A.1). $\square$

Remark A.9. The property (iv) of Lemma A.8 equivalently means that $\widehat{f}$ is lower semi-continuous for every lower semi-continuous function $f \colon Y \to [0, \infty)$. Here $\widehat{f}$ is defined as

$$\widehat{f}(x) := \sup f\left(\pi^{-1}\left(\overline{\{x\}}\right)\right).$$

(Indeed: If $t \in [0, \infty)$, then $\widehat{f}^{-1}[0, t] = F_G$ for $G := f^{-1}[0, t]$. One can take in place of f the characteristic function of any open subset $U = Y \setminus G$ of Y.)

Lemma A.10. *A continuous map $\pi \colon Y \to X$ is pseudo-epimorphic if and only if the map $\Theta \colon U \mapsto U \cap \pi(Y)$ defines a lattice isomorphism from $\mathbb{O}(X)$ onto $\mathbb{O}(\pi(Y))$.*

Proof. π is pseudo-epimorphic, if and only if, $U \setminus V$ contains a point of $\pi(Y)$ for all open subsets $V \subset U$ of X with $V \neq U$. (Indeed, let $F := X \setminus V$, $G := X \setminus U$, then $G \subset F$ are closed, $G \neq F$ and $F \setminus G = U \setminus V$. If π is pseudo-epimorphic, then F is the closure of $F \cap \pi(Y)$ and $F \setminus G$ must contain a point of $\pi(Y)$. Conversely, if π is not pseudo-epimorphic, then there F is a closed subset of X such that $G := \overline{F \cap \pi(Y)} \subset F$ is not equal to F. Then $U \setminus V = F \setminus G$ does not contain a point of $\pi(Y)$.)

Clearly Θ is a lattice epimorphism.

Let $\Theta(U_1) = \Theta(U_2)$ then $\Theta(U_1 \cap U_2) = \Theta(U_1 \cup U_2)$. Suppose that $U_1 \neq U_2$. Then $(U_1 \cup U_2) \setminus (U_1 \cap U_2)$ is not empty and does not contain a point of $\pi(Y)$. Thus Θ must be an isomorphism if π is pseudo-epimorphic.

Conversely suppose that Θ is a lattice isomorphism and $V \subset U$ with $V \neq U$ then $V \cap \pi(Y) = \Theta(V) \neq \Theta(U) = U \cap \pi(Y)$. It yields that $U \setminus V$ contains a point of $\pi(Y)$. Hence, π is pseudo-epimorphic. $\square$

Proposition A.11. *Suppose that X and P are point-complete* T_0*-spaces. Then there is a one-to-one correspondence between maps $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(P)$ with properties*

(I_0) $\Psi(X) = P$ *and* $\Psi(\emptyset) = \emptyset$,

(II_0) $\Psi(U \cap V) = \Psi(U) \cap \Psi(V)$ *for all open subsets* $U, V \subset X$,

(III) $\Psi\left(\left(\bigcup_\alpha U_\alpha\right)\right) = \bigcup_\alpha \Psi\left(U_\alpha\right)$ *for every family of open subsets* $U_\alpha \subset X$

and continuous maps $\pi\colon P \to X$ given by

$$\Psi(U) := \pi^{-1}(U)$$

and

$$\pi(p) := x$$

where x is defined by Lemma A.5.

π is pseudo-open and pseudo-epimorphic if and only if Ψ satisfies in addition properties (II) and (IV) of Definition 1.1, i.e. if and only if Ψ is a lattice monomorphism from $\mathbb{O}(X)$ into $\mathbb{O}(P)$ that respects l.u.b. and g.l.b. and satisfies (I_0).

Proof. Suppose that $\pi\colon P \to X$ is continuous. Then $\Psi(U) := \pi^{-1}U$ clearly satisfies (I_0), (II_0) and (III).

Suppose that π_1 is another map with $\Psi(U) = \pi_1^{-1}(U)$. $P \setminus \pi_1^{-1}(U) = P \setminus \pi^{-1}(U)$ for $U := X \setminus \overline{\{\pi(p)\}}$, implies

$$\pi_1\left(\pi^{-1}\left(\overline{\{\pi(p)\}}\right)\right) = \pi_1(P) \cap \overline{\{\pi(p)\}}.$$

In particular, $\pi_1(p)$ is in $\overline{\{\pi(p)\}}$. Similarly, $\pi(p) \in \overline{\{\pi_1(p)\}}$. Since X is T_0, $\pi_1(p) = \pi(p)$.

Suppose Ψ with (I_0), (II_0) and (III) is given, and let π be as in Lemma A.5.

By Lemma A.6 $\pi^{-1}U = \Psi(U)$ holds for $U \in \mathbb{O}(X)$. In particular $\pi\colon P \to X$ is continuous.

Suppose that π is in addition a pseudo-open and pseudo-epimorphic map. Then the inclusion map from $\pi(P)$ into X induces an isomorphism from $\mathbb{O}(X)$ onto $\mathbb{O}(\pi(P))$ by Lemma A.10. On the other hand, the continuous epimorphism from P onto $\pi(P)$ is still a pseudo-open map, because Definition 1.3(ii) refers only to the pseudo-graph R_π and the open subsets of $\pi(P)$. Let V be an R_π-invariant open subset of P. Then $\pi(V)$ is an open subset of $\pi(P)$ by Definition 1.3. For $p \in \pi^{-1}(\pi(V))$ there is $q \in V$ such that $\pi(q) = \pi(p)$, in particular, $(p, q) \in R_\pi$, thus $p \in V$ and $V = \pi^{-1}(\pi(V)$. This shows that $\pi\colon P \to \pi(P)$ satisfies the conditions of Lemma A.8(iii). Ψ satisfies (I)–(IV) of Definition 1.1 by Lemma A.8(ii), because $\Psi(U) = \pi^{-1}(U) = \pi^{-1}(U \cap \pi(P))$, and $U \mapsto U \cap \pi(P)$ is a lattice isomorphism.

Conversely, if Ψ satisfies (I)–(IV) of Definition 1.1, then the lattice monomorphism Ψ factorizes through the lattice epimorphism $\Psi_0\colon U \mapsto U \cap \pi(P)$. Thus Ψ_0 is a lattice isomorphism, and π is a pseudo-epimorphism by Lemma A.10. It follows that the continuous epimorphism π from P onto $\pi(P)$ satisfies (ii) of Lemma A.10 (by condition (II) of Ψ). Thus π is also pseudo-open. $\square$

Corollary A.12. *Suppose that X and Y are point-complete* T_0*-spaces. If Ψ is an isomorphism from $\mathbb{O}(X)$ onto $\mathbb{O}(Y)$, then there is a unique homeomorphism π from Y onto X such that $\Psi(U) = \pi^{-1}U$ for $U \in \mathbb{O}(X)$.*

Example A.13. Let $(0,1]_{\mathrm{lsc}}$ denote the half-open interval $(0,1]$ with topology

$$\mathbb{O}((0,1]_{\mathrm{lsc}}) := \{\emptyset, (t,1] : \ t \in [0,1)\}$$

The continuous epimorphism

$$t \in (0,1] \mapsto t \in (0,1]_{\mathrm{lsc}}$$

is pseudo-open but is not open.

Lemma A.14. *If C is a regular* C^**-subalgebra of B (cf. Definition 1.2), then the map*

$$J \in \mathbb{I}(B) \mapsto J \cap C \in \mathbb{I}(C)$$

defines naturally a map $\Psi\colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathbb{O}(\mathrm{Prim}(C))$ that satisfies properties (I)–(IV) of Definition 1.1.

Proof. By the correspondence between closed ideals and open subsets of primitive ideal spaces, (II) is equivalent to $(\bigcap_\alpha J_\alpha) \cap C = \bigcap_\alpha (J_\alpha \cap C)$, and (IV) means that $J_1 \cap C = J_2 \cap C$ implies $J_1 = J_2$ (and follows from (ii) of Definition 1.2). (I) holds for Ψ, because $\{0\} \cap C = \{0\}$ and $J \cap C = C$ implies $J = B$. Ψ satisfies (III), because $(J_1 + J_2) \cap C = (J_1 \cap C) + (J_2 \cap C)$ for all J_1, J_2 in $\mathbb{I}(B)$ by (i) of Definition 1.2, and because $\left(\overline{\bigcup_\alpha J_\alpha}\right) \cap C = \overline{\bigcup_\alpha (J_\alpha \cap C)}$ holds for every upward directed family $\{J_\alpha\}_\alpha$ in $\mathbb{I}(B)$. The latter follows from $\operatorname{dist}(c, \bigcup_\alpha J_\alpha) = \inf_\alpha \operatorname{dist}(c, J_\alpha)$ and $\operatorname{dist}(c, J_\alpha) = \operatorname{dist}(c, J_\alpha \cap C)$ for $c \in C$. □

Lemma A.15. *Suppose that A is a separable* C*-algebra and that Y is a locally compact Hausdorff space. Moreover suppose that*

$$\Psi \colon \mathbb{O}(Y) \to \mathbb{I}(A) \cong \mathbb{O}(\operatorname{Prim}(A))$$

is an order preserving map. Let $K_p \subset \mathcal{Q}(A)$ denote the convex set of quasi-states ξ on A with $\xi(\Psi(Y \setminus \{p\})) = \{0\}$.

Then, Ψ satisfies property (II) of Definition 1.1, if and only if, the map $(p, \xi) \mapsto p \in Y$ is an open map from $R := \{(p, \xi) : \ \xi \in K_p\} \subset Y \times \mathcal{Q}(A)$ onto Y.

If Ψ satisfies (II) of Definition 1.1, then

(i) every continuous selection $p \mapsto \xi_p \in K_p$ defines a completely positive contraction $T \colon A \to C_b(Y)$ with $T(\Psi(U))\, C_0(Y) \subset C_0(U) \subset C_0(Y)$ for all $U \in \mathbb{O}(Y)$, and,

(ii) for every $W \in \mathbb{O}(Y)$ and every $a \in A \setminus \Psi(W)$, there exists a completely positive contraction $T \colon A \to C_0(Y)$ with $T(a) \notin C_0(W)$ and $T(\Psi(U)) \subset C_0(U)$ for all $U \in \mathbb{O}(Y)$.

Proof. Recall that the boundary $\partial\mathcal{Q}(A)$ of $\mathcal{Q}(A)$ is just $\{0\} \cup P(A)$, where $P(A)$ denotes the pure states on A. Further recall that $\pi \colon \xi \in P(A) \mapsto J_\xi \in \operatorname{Prim}(A)$ is an open and continuous epimorphism from $P(A)$ onto $\operatorname{Prim}(A)$ (cf. [6, thm. 3.4.11], [23, thm. 4.3.3]). Here J_ξ is the kernel of the GNS representation corresponding to ξ, i.e. $b \in J_\xi$ if and only if $\xi(AbA) = \{0\}$.

Let $\Psi'(F) := \{0\} \cup \pi^{-1}(\mathsf{h}(\Psi(Y \setminus F)))$ for $F \in \mathbb{F}(Y)$, then it is straight forward to check that $\Psi \colon \mathbb{O}(Y) \to \mathbb{I}(A)$ fulfills condition (II) of Definition 1.1 if and only if $\Psi' \colon \mathbb{F}(Y) \to \mathbb{F}(\{0\} \cup P(A))$ satisfies

(II′) $\Psi'\left(\overline{\bigcup_\alpha F_\alpha}\right) = \overline{\bigcup_\alpha \Psi'(F_\alpha)}$ for every (non-empty) family $\{F_\alpha\}$ of closed subsets of Y.

It is easy to see that (II′) is equivalent to $\Psi'(F) = \overline{\bigcup_{p\in F} \Psi'(\{p\})}$ for every closed subset $F \neq \emptyset$ of Y. Since $K_p \cong \mathcal{Q}(A/\Psi(Y \setminus \{p\}))$ is a split face of $\mathcal{Q}(A)$ for $p \in Y$, it holds:

$$\Psi'(\{p\}) = K_p \cap (\{0\} \cup P(A)) = K_p \cap \partial\mathcal{Q}(A) = \partial K_p\,.$$

By Lemma A.7, $(p, \xi) \in R \mapsto p \in Y$ is open, if and only if,

$$(p, \xi) \in R \cap (Y \times (\{0\} \cup P(A))) \to p \in Y$$

is open, if and only if, $\Psi'(F) = \overline{\bigcup_{p\in F} \Psi'(\{p\})}$ for every closed subset F of Y.

Ad (i): The function $p \mapsto T(a)(p) := \xi_p(a)$ is continuous on Y for every $a \in A$, because the selection is weakly continuous. $T\colon A \to C_b(Y)$ is completely positive, because $a \mapsto T(a)(p) = \xi_p(a)$ is completely positive for every $p \in Y$.

If $a \in \Psi(U)$ and $q \in Y \setminus U$ then $\xi_q(a) = 0$, because $\Psi(U) \subset \Psi(Y \setminus \{q\})$ and $\xi_q \in K_q$. Thus $p \mapsto T(a)(p)f(p)$ is in $C_0(U)$ for $a \in \Psi(U)$ and $f \in C_0(Y) \cong C$.

Ad (ii): By property (II), $\Psi(W) = \bigcap_{p\in Y\setminus W} \Psi(Y \setminus \{p\})$. Thus, there is $q \in Y \setminus W$ with $a \notin \Psi(Y \setminus \{q\})$, and there is a (pure) state $\chi \in K_q$ with $\chi(a) \neq 0$.

Since $(p, \xi) \mapsto p$ is an open map from R onto Y,

$$p \in Y \mapsto K_p \subset \mathcal{Q}(A)$$

is a lower semi-continuous family (with respect to the $\sigma(A^*, A)$-topology on $\mathcal{Q}(A)$). It follows by a selection theorem of Michael [22] that there is a continuous selection $p \mapsto \xi_p \in K_p$ with $\xi_q = \chi$. Let $f \in C_0(Y) \cong C$ a function with $0 \le f \le 1$ and $f(q) = 1$. Then $T\colon A \to C_0(Y)$ with $T(a)(p) = f(p)\xi_p(a)$ satisfies $T(a)(q) = \chi(a) \neq 0$, i.e. $T(a) \notin C_0(W)$. Hence, by (i), T is as desired in (ii). □

A.3 Hilbert C*-modules

Now we recall the definitions of Hilbert C*-modules, Hilbert A-bi-modules, its tensor products and some basic results on its module homomorphisms (cf. [21] for details).

Definition A.16. A *pre-Hilbert A-module* is a right A-module E over a C*-algebra A equipped with an A-valued sesquilinear form $\langle \cdot, \cdot \rangle : E \times E \to A$ that is A–linear in the second variable such that $\langle e, e \rangle \geq 0$ for all $e \in E$ and $\langle e, e \rangle = 0$ implies $e = 0$.

E becomes a normed vector space with norm $\|e\| := \sqrt{\| \langle e, e \rangle \|}$. A pre-Hilbert A-module which is complete with respect to the norm induced by $\langle ., . \rangle$ is called *Hilbert A-module*.

E is *full* if the span of $\langle E, E \rangle$ is dense in A, i.e., if A is the closure of the linear span of $\{\langle e, f \rangle \;:\; e, f \in E\}$.

A Hilbert A-module E that is the closure of the A-linear span (i.e. of the set of finite sums $\sum e_j a_j$) of a countable subset $\{e_1, e_2, \ldots\}$ of E is called *countably generated* over E.

A map $\iota \colon E \to F$ from E onto another Hilbert A-module F is an *isomorphism* of Hilbert modules, if ι is an epimorphism of A-modules and satisfies $\langle \iota(e_1), \iota(e_2) \rangle = \langle e_1, e_2 \rangle$ for $e_2, e_2 \in E$.

To keep notation simple, it is useful to note that an *isometric* A-module map ι from E to F automatically satisfies $\langle \iota(e_1), \iota(e_2) \rangle = \langle e_1, e_2 \rangle$. This follows by polar decomposition from the fact that for $a, b \in A_+$ holds $a \leq b$ if and only if $\|(b+t)^{-1/2} a (b+t)^{-1/2}\| \leq 1$ for $t \in (0, \infty)$.

Example A.17.
(i) A C*-algebra A itself becomes a (right) Hilbert A-module A_A with the algebra-multiplication from the right and the scalar-product $\langle a, b \rangle := a^* b$. More generally, every closed right ideal R of A becomes a Hilbert A-module R_A with this A-valued sesquilinear form.

(ii) Sequences $(a_n) \in A^{\mathbb{N}}$ build a right A-module with multiplication of the a_n from the right by elements of A. By restricting to sequences where the series $\sum_{n=1}^{n} a_n^* a_n$ converges in A and taking the scalar-product $\langle (a_n), (b_n) \rangle := \sum_{n=1}^{\infty} a_n^* b_n$ we get a Hilbert A-module, which we denote by $\mathcal{H}_A$. If A is stable, then there is an isometric A-module isomorphism from $\mathcal{H}_A$ onto the Hilbert A-module A of (i) (cf. Remark 2.5).

(iii) More generally if $\{E_\xi\}_{\xi \in X}$ is any family of Hilbert A-modules, then one can define a Hilbert A-module $\bigoplus_{\xi \in X} E_\xi$ as the set of all maps $e \colon X \to \bigcup_{\xi \in X} E_\xi$ with $e(\xi) \in E_\xi$ and $\sum_\xi e(\xi)^* e(\xi)$ convergent in A, together with the obvious right A-module structure and obvious sesquilinear form.

(iv) By a result of Kasparov [13] (cf. [1, thm. 13.6.3], [21, thm. 6.2]) for every countably generated Hilbert A-module E holds $E \oplus \mathcal{H}_A \cong \mathcal{H}_A$ (by an isometric A-module isomorphism), i.e., every countably generated Hilbert A-module is the range of a self-adjoint projection on $\mathcal{H}_A$.

Definition A.18. If E is a Hilbert A-module then $\mathcal{L}(E)$ denotes the set of adjoint-able maps over E, i.e. maps $T\colon E \to E$ for which there is a map $T^*\colon E \to E$ with

$$\langle e, Tf\rangle = \langle T^*e, f\rangle \quad \text{for all } e, f \in E.$$

$\mathbb{K}(E)$ is defined as the closed linear span of the set of maps over E given for $x, y \in E$ by

$$\theta_{xy}\colon E \to E, e \mapsto x\langle y, e\rangle .$$

Adjoint-able maps from (all of) E into E (with adjoint maps defined on all of E) are automatically A-module homomorphisms, and are bounded (by an application of the Banach-Steinhaus theorem). Furthermore, $\mathcal{L}(E)$ is a unital C*-algebra. It holds $\theta_{xy}^* = \theta_{yx}$, and $\mathbb{K}(E)$ is an essential closed ideal of $\mathcal{L}(E)$. The natural *-monomorphism from $\mathcal{L}(E)$ to $\mathcal{M}(\mathbb{K}(E))$ is an isomorphism from $\mathcal{L}(E)$ onto $\mathcal{M}(\mathbb{K}(E))$ that turns strong* topology on (bounded parts of) $\mathcal{L}(E)$ to the strict topology on (bounded parts of) $\mathcal{M}(\mathbb{K}(E))$. Moreover, $\mathbb{K}(E)$ is strongly Morita equivalent to A in a natural way.

Example A.19. For Examples A.17(i) the sets are well-known C*-algebras: $\theta_{xy}(a) = xy^*a$ and by that $\mathbb{K}(A_A) = A$. The adjoint-able operators are the multipliers of A, $\mathcal{L}(A_A) = \mathcal{M}(A)$.

In the case of a closed right ideal R of A, $\mathbb{K}(R_A) = R^* \cap R = RR^*$ (a hereditary C*-subalgebra of A), and $\mathcal{L}(R_A) = \mathcal{M}(R^* \cap R)$.

Given a Hilbert B-module E and a *-representation $h\colon B \to \mathcal{L}(F)$ of B on a Hilbert A-module F, one can define a Hilbert A-module $E \otimes_h F$, the *(interior) tensor product*:

A right A-module structure $x \cdot b$ and an A-valued sesquilinear form $\langle x, y\rangle$ on the *algebraic* vector space tensor product $E \odot F$ of E and F are given by

$$\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle := \langle f_1, h(\langle e_1, e_2\rangle) f_2\rangle$$

and $(e \otimes f)b := e \otimes (fb)$. It can be shown that the linear subspace L generated by elements of the form $ea \otimes f - e \otimes h(a)f$ is equal to the set $\{x \in E \odot F : \; < x, x >= 0\}$ (cf. proof of [21, prop. 4.5]). $E \otimes_h F$ is defined as the completion of the quotient $(E \odot F)/L$. Below we use the notation

$e \otimes_h f$ (or $e \otimes_B f$ if no confusion can arise) for the element $(e \otimes f) + L$ in the quotient $(E \odot F)/L \subset E \otimes_h F$.

Remark A.20. (i) There is a natural unital *-homomorphism η from $\mathcal{L}(E)$ into $\mathcal{L}(E \otimes_h F)$ given on elementary tensors by $\eta(S)\,(e \otimes_h f) := S(e) \otimes_h f$ for $S \in \mathcal{L}(E)$, because the map $T \mapsto T \otimes \mathrm{id} \in \mathrm{Lin}(E \odot F)$ is multiplicative and satisfies $\langle x, T \otimes \mathrm{id}(y)\rangle = \langle T^* \otimes \mathrm{id}(x), y\rangle$. (It gives that $\|T \otimes \mathrm{id}(y)\| \leq \|T\|$ if applied to $S := (\|T\|^2 - T^*T)^{1/2}$, thus $T \otimes \mathrm{id}(L) \subset L$, and $[T]_L(x + L) := T \otimes \mathrm{id}(x) + L$ extends to an adjoint-able operator $\eta(T)$ with norm $\leq \|T\|$.)

(ii) Thus, if the Hilbert B-module has a left C-module structure given by a *-homomorphism $k \colon C \to \mathcal{L}(E)$ then the Hilbert A-module $E \otimes_h F$ has a natural left C-module structure given by $\eta \circ k \colon C \to \mathcal{L}(E \otimes_h F)$.

(iii) Another map is given by $L_e \colon f \in F \to e \otimes f \in E \otimes_h F$ for $e \in F$. It is easy to see that L_e is a A-module map, $\|L_e\| \leq \|e\|$ and has the adjoint $(L_e)^* \colon E \otimes_h F \to F$ given elementary tensors by $(L_e)^*(e' \otimes_h f) = h(\langle e, e'\rangle)f$.

Example A.21. If $h \colon B \to \mathcal{M}(A)$ is a *-homomorphism, then there is a natural isometric Hilbert A-module isomorphism I from $B \otimes_h A$ onto the closed right ideal $R_h := \overline{\mathrm{span}\,(h(B)A)}$. I is given by $I\,(b \otimes_h a) := h(b)a$.

$\mathcal{L}(B \otimes_h A) \cong \mathcal{M}(R_h^* \cap R_h)$ (cf. Example A.19), and $I\eta(\,\cdot\,)I^{-1}$ is the unital strictly continuous *-homomorphism from $\mathcal{M}(B)$ into $\mathcal{M}(R_h^* \cap R_h)$ that extends the non-degenerate *-homomorphism $\iota \circ h \colon B \to \mathcal{M}(R_h^* \cap R_h)$, where ι is the restriction *-homomorphism from $\{t \in \mathcal{M}(A) : \; tA + t^*A \subset R_h\}$ into $\mathcal{M}(R_h^* \cap R_h)$.

$R_h = A$ if and only if $h \colon B \to \mathcal{M}(A)$ is non-degenerate. Then $B \otimes_h A \cong A$, and under this isomorphism L_e becomes $h(e)$ and $(L_e)^* = h(e^*)$, and $\eta \colon B \to \mathcal{L}(B \otimes_h A)$ becomes the natural extension of h to a unital strictly continuous map $\mathcal{M}(h) : \mathcal{M}(B) \to \mathcal{M}(A)$. (We denote $\mathcal{M}(h)$ also by h to keep notation simple.)

Definition A.22. A *Hilbert (B,A)-bi-module* is a right Hilbert A-module E together with a left B-module structure given by a *-homomorphism $h \colon B \to \mathcal{L}(E)$. I.e. let $h \colon B \to \mathcal{L}(E)$ be a *-homomorphism, then E is an (B,A)-bi-module with the left multiplication given by $a \cdot e := h(a)e$.

E is *full* if E is full as (right) Hilbert A-module.

E is *non-degenerate* if the linear span of $h(B)E$ is dense in E, i.e. if $h(B)E = E$ (by Cohen factorization theorem).

We denote by $\mathcal{H}(A, h)$ the non-degenerate Hilbert (A, A)-bi-module that is given by the Hilbert A-module $E = A_A$ of Example A.17(i) and a *non-degenerate* *-homomorphism $h\colon A \to \mathcal{L}(E) = \mathcal{M}(A)$.

Remark A.23. Suppose that $h_i\colon A \to \mathcal{M}(A)$ ($i = 1, 2$) are *non-degenerate* *-homomorphisms. Then h_i uniquely extends to a strictly continuous unital *-homomorphism $h_i\colon \mathcal{M}(A) \to \mathcal{M}(A)$ (cf. Example A.21). Since $h_2 \circ h_1$ is strictly continuous and unital, we get that $(h_2 \circ h_1|A) : A \to \mathcal{M}(A)$ is non-degenerate. Thus we can define a non-degenerate *-homomorphism $h_2 \circ h_1\colon A \to \mathcal{M}(A)$ as the restriction of the strictly continuous unital *-homomorphism $h_2 \circ h_1$ from $\mathcal{M}(A)$ into $\mathcal{M}(A)$.

We have seen in Example A.21, that there is a Hilbert A-module isomorphism I from $A \otimes_{h_2} A$ onto the (right) Hilbert A-module A of Example A.17(i). I is given by $I(a \otimes_{h_2} b) := h_2(a)b$. The left A-module structure on $A \otimes_{h_2} A$ is defined by $a \cdot (b \otimes_{h_2} c) := \eta(h_1(a))(b \otimes_{h_2} c) = (h_1(a)b) \otimes_{h_2} c$ for $a, b, c \in A$.

Thus, $I(a \cdot (b \otimes_{h_2} c)) = h_2(h_1(a)b)\, c = h_2 \circ h_1(a) I(b \otimes_{h_2} c)$, and I is also a left A-module isomorphism from the left A-module $A \otimes_{h_2} A$ onto the left A-module A given by $(h_2 \circ h_1) : A \to \mathcal{M}(A)$.

The n-fold tensor product $E_1 \otimes_{h_2} E_2 \otimes_{h_3} \cdots \otimes_{h_n} E_n$ of the Hilbert A-bi-modules E_i given by non-degenerate $h_i\colon A \to \mathcal{M}(A)$ is defined inductively by

$$E_1 \otimes_{h_2} E_2 \otimes_{h_3} \cdots \otimes_{h_n} E_n := \big(E_1 \otimes_{h_2} E_2 \otimes_{h_3} \cdots \otimes_{h_{n-1}} E_{n-1}\big) \otimes_{h_n} E_n\,.$$

By induction, it is isomorphic to the Hilbert A-module A of Example A.17(i) with left A-module structure given by $h_n \circ h_{n-1} \circ \cdots \circ h_1\colon A \to \mathcal{M}(A)$, and there is a natural Hilbert A-module isomorphism

$$E_1 \otimes_{h_n \circ \cdots \circ h_2} (E_2 \otimes_{h_3} \cdots \otimes_{h_n} E_n) \cong E_1 \otimes_{h_2} E_2 \otimes_{h_3} \cdots \otimes_{h_n} E_n\,.$$

Some Hilbert bi-modules are related to completely positive maps:

Lemma A.24. *For every completely positive map $T\colon A \to B$ there are a Hilbert B-module E^T and a *-homomorphism $h^T\colon A \to \mathcal{L}\left(E^T\right)$, such that h^T is non-degenerate and (E^T, h^T) satisfies:*

A completely positive map $V\colon A \to B$ can be approximated in point-norm by maps $V_e\colon A \to B$ given by $V_e(a) := \left\langle e, h^T(a)e\right\rangle$, if and only if, V can be approximated by by completely positive maps $T_{r,c}\colon A \to B$ given by

$T_{r,c}(a) := c^* T \otimes \mathrm{id}_n(r^* a r) c$ *for a row-matrix* $r \in M_{1,n}(A)$ *and a column-matrix* $c \in M_{n,1}(B)$*,* $n \in \mathbb{N}$*.*

E^T *can be taken countably generated over* B *if* A *is separable.*

Proof. The algebraic vector space tensor product $A \odot B$ has natural left A-module and right B-module structures. Define a B-valued sesquilinear form $\beta(x, y)$ on elementary tensors by

$$\beta(a_1 \otimes b_1, a_2 \otimes b_2) := b_1^* T(a_1^* a_2) b_2 \,.$$

It is B-linear in the second variable, satisfies $\beta(x, x) \geq 0$, $\beta(a \cdot x, a \cdot x) \leq \|a\|^2 \beta(x, x)$, and $\beta(a^* \cdot x, y) = \beta(x, a \cdot y)$ for all $x, y \in A \odot B$, $a \in A$.

Thus, the subspace $L := \{x \in A \odot B : \ \beta(x, x) = 0\}$ is A- and B-invariant, and β defines on $(A \odot B)/L$ a B-valued scalar product $\langle x, y \rangle$,which defines a pre-Hilbert B-module. The completion E^T of $(A \odot B)/L$ and $\langle \cdot, \cdot \rangle$ is a Hilbert B-module with a *-homomorphism $h^T \colon A \to \mathcal{L}\left(E^T\right)$.

By Cohen factorization, for every $x \in A \odot B$ there is $y \in A \odot B$ and $a \in A$ with $(a \otimes 1_{\mathcal{M}(B)}) y = x$. It implies that $h^T(A) E^T$ is dense in E^T.

The characterization of c.p. maps which can be approximated by maps V_e follows immediately from the definition of E^T.

If A is separable, let $c \in C_+$ a strictly positive element of the separable C*-subalgebra C of B that is generated by $V(A)$, and let $\{a_1, a_2, \ldots\}$ a dense sequence in A. Then the B-module generated by $\{(a_1 \otimes c) + L, (a_2 \otimes c) + L, \ldots\}$ is dense in E^T. □

A.4 Crossed products by $\mathbb{Z}$

Remark A.25. Suppose that $\varphi \colon E \to F$ a *-homomorphism, $\alpha \in \mathrm{Aut}\,(E)$, $\beta \in \mathrm{Aut}\,(F)$ with $\varphi \circ \alpha = \beta \circ \varphi$. Recall the following elementary properties of crossed products by $\mathbb{Z}$:

(i) The universal crossed product $E \rtimes_\alpha \mathbb{Z}$ is the same as the reduced crossed product $E \rtimes_{\alpha,r} \mathbb{Z}$.

$E \rtimes_{\alpha,r} \mathbb{Z}$ is the image of the *-representation d_{red} from $E \rtimes_\alpha \mathbb{Z}$ into the von-Neumann algebra $M \overline{\otimes} \mathcal{L}\,(\ell_2(\mathbb{Z}))$ given by

$$d_{\mathrm{red}}(e) := (\ldots, \alpha^{-1}(e), e, \alpha(e), \alpha^2(e), \ldots) \in \ell_\infty(E^{**}) \subset M \overline{\otimes} \mathcal{L}\,(\ell_2(\mathbb{Z}))$$

for $e \in E$ and $\mathcal{M}\,(d_{\mathrm{red}})\,(U_0) := 1 \otimes S_1$. Here M is any W*-algebra that contains A as a weakly dense C*-subalgebra, and $S_1(f)(n) := f(n+1)$

for $f \in \ell_2(\mathbb{Z})$ and U_0 is the canonical generator of $\mathbb{Z}$ in $\mathcal{M}(E \rtimes_\alpha \mathbb{Z})$ with $U_0 e U_0^* = \alpha(e)$ for $e \in E$ (see (v) below).

(ii) E is naturally isomorphic to a C*-subalgebra of $E \rtimes_\alpha \mathbb{Z}$.

(iii) φ defines a *-homomorphism $\varphi \rtimes \mathbb{Z} \colon E \rtimes_\alpha \mathbb{Z} \to F \rtimes_\beta \mathbb{Z}$ with

$$\varphi(E) = F \cap (\varphi \rtimes \mathbb{Z}(E \rtimes_\alpha \mathbb{Z}))\,.$$

$\varphi \rtimes \mathbb{Z}$ is a monomorphism if φ is a monomorphism.

(iv) If $\varphi(E)$ is an ideal of F, then $E \rtimes_\alpha \mathbb{Z}$ maps onto an ideal of $F \rtimes_\beta \mathbb{Z}$.

(v) The non-degenerate *-representations $\varrho \colon E \rtimes_\alpha \mathbb{Z} \to \mathcal{L}(H)$ are in one-to-one correspondence to covariant representations (ϱ_E, U, H) (where $\varrho_E \colon E \mapsto \mathcal{L}(H)$ is a non-degenerate *-representation and $U \in \mathcal{L}(H)$ is a unitary with $\varrho_E(\alpha(e)) = U\varrho_E(e)U^*$ for $e \in E$). The image of ϱ is generated by $U\varrho_E(E)$ as C*-algebra.

(vi) The crossed product $(E, \alpha) \mapsto E \rtimes_\alpha \mathbb{Z}$ defines in a natural manner an exact functor from the category of dynamical C*-systems into the category of C*-algebras (as follows from (i), (iii) and (iv)).

Lemma A.26. *Let $\alpha \in \mathrm{Aut}(B)$, B unital. If, for $\theta \in \mathbb{C}$ with $|\theta| = 1$, there is a unitary $v(\theta)$ in the center of B such that $\theta\alpha(v(\theta)) = v(\theta)$, then every ideal I of $B \rtimes_\alpha \mathbb{Z}$ is invariant under the dual action $\hat{\alpha}$ of $\mathbb{T} = S^1$ on $B \rtimes_\alpha \mathbb{Z}$ and, thus, is determined by its intersection $I \cap B$ with B, i.e. I is the natural image of $(I \cap B) \rtimes_\alpha \mathbb{Z}$ in $B \rtimes_\alpha \mathbb{Z}$.*

Proof. The ideals I of $B \rtimes_\alpha \mathbb{Z}$ are invariant under the dual action $\widehat{\alpha}$ of $\mathbb{T} = S^1$ on $B \rtimes_\alpha \mathbb{Z}$, because $\widehat{\alpha}(\theta) = v(\theta)(\,.\,)v(\theta)^*$ for $\theta \in \mathbb{T}$.

There is no ideal $\neq 0$ on a crossed-product $B \rtimes_\alpha \mathbb{Z}$ that is invariant under the dual action of $\mathbb{T}$ on $B \rtimes_\alpha \mathbb{Z}$ *and* is orthogonal to B, because the integral over the dual action is a *faithful* conditional expectation from $B \rtimes_\alpha \mathbb{Z}$ onto B. This remains true for every quotient of B by an α-invariant ideal J of B. Thus, every non-zero closed ideal of

$$(B \rtimes_\alpha \mathbb{Z})/(J \rtimes_{\alpha|J} \mathbb{Z}) \cong (B/J) \rtimes_{[\alpha]} \mathbb{Z}$$

intersects B/J. (See Remark A.25(vi) for the isomorphism.)

If I is a closed ideal of $B \rtimes_\alpha \mathbb{Z}$ then $J := I \cap B$ is α-invariant, the natural image of $J \rtimes \mathbb{Z}$ is contained in I and the natural image of I in $(B/J) \rtimes \mathbb{Z}$ has zero intersection with B/J, thus I is the natural image of $(I \cap B) \rtimes \mathbb{Z}$. □

For the following Lemma A.27 recall that:

(i) $\ell_\infty(\mathcal{M}(A)) \subset \mathcal{M}(\mathcal{M}(A) \otimes \mathbb{K}) \subset \mathcal{M}(A \otimes \mathbb{K}) \cong \mathcal{L}(\mathcal{H}_A)$ where $\mathcal{H}_A$ is as in Example A.17(ii),

(ii) $\ell_\infty(\mathcal{M}(A)) \cap (\mathcal{M}(A) \otimes \mathbb{K}) = c_0(\mathcal{M}(A))$,

(iii) $\mathcal{T} \in 1 \otimes \mathcal{L}(\ell_2) = 1_{\mathcal{M}(A)} \otimes \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(\mathcal{M}(A) \otimes \mathbb{K})$ where $\mathcal{T} := 1 \otimes \mathcal{T}_0$ with $\mathcal{T}_0 \in \mathcal{L}(\ell_2)$ is the Toeplitz operator (= forward shift) on $\ell_2 := \ell_2(\mathbb{N})$.

(iv) Let $Q^s(\mathcal{M}(A)) := \mathcal{M}(\mathcal{M}(A) \otimes \mathbb{K}) / (\mathcal{M}(A) \otimes \mathbb{K})$ the *stable corona* of $\mathcal{M}(A)$, and

(v) let $B := \ell_\infty(\mathcal{M}(A))/c_0(\mathcal{M}(A))$. By (ii), there is a natural unital *-monomorphism $\epsilon\colon B \hookrightarrow Q^s(\mathcal{M}(A))$.

(vi) $\sigma \in \mathrm{Aut}(B)$ denote the automorphism of B that is induced by the forward shift on $\ell_\infty(\mathcal{M}(A)) := \ell_\infty(\mathbb{N}, \mathcal{M}(A))$.

Lemma A.27.

$$\mathcal{M}(A) \otimes \mathbb{K} \subset \mathrm{C}^*(\ell_\infty(\mathcal{M}(A)), \mathcal{T}) \subset \mathcal{M}(\mathcal{M}(A) \otimes \mathbb{K}),$$

and $\mathrm{C}^*(\ell_\infty(\mathcal{M}(A)), \mathcal{T}) / (\mathcal{M}(A) \otimes \mathbb{K})$ *is naturally isomorphic to* $B \rtimes_\sigma \mathbb{Z}$ *by the *-homomorphism that is defined by the covariant representation* (ϵ, U) *of* (B, σ) *into* $Q^s(\mathcal{M}(A))$.

Here $B := \ell_\infty(\mathcal{M}(A)) / c_0(\mathcal{M}(A))$, σ *is induced by the forward shift on* $\ell_\infty(\mathcal{M}(A)) := \ell_\infty(\mathbb{N}, \mathcal{M}(A))$, *and* $U := \mathcal{T} + \mathcal{M}(A) \otimes \mathbb{K}$.

Proof. Since $1 \otimes \mathbb{K} = (1 \otimes \mathcal{L}(\ell_2)) \cap (\mathcal{M}(A) \otimes \mathbb{K})$ and $1 - \mathcal{T}_0\mathcal{T}_0^* \in \mathbb{K}$, the image $U := \mathcal{T} + \mathcal{M}(A) \otimes \mathbb{K}$ of $\mathcal{T} = 1 \otimes \mathcal{T}_0$ in $Q^s(\mathcal{M}(A))$ is unitary. A straight calculation shows that $a \mapsto \mathcal{T} a \mathcal{T}^*$ is the forward shift on $\ell_\infty(\mathcal{M}(A))$, thus $\epsilon(\sigma(b)) = U\epsilon(b)U^*$ for $b \in B$.

$\mathcal{M}(A) \otimes \mathbb{K}$ is contained in $\mathrm{C}^*(\ell_\infty(\mathcal{M}(A)), \mathcal{T})$, because $\mathrm{C}^*(\mathcal{T}_0 \cdot c_0) = \mathbb{K}$ and $\mathrm{C}^*(\mathcal{T} c_0(\mathcal{M}(A))) = \mathcal{M}(A) \otimes \mathrm{C}^*(\mathcal{T}_0 \cdot c_0)$.

$$\mathrm{C}^*(\ell_\infty(\mathcal{M}(A)), \mathcal{T}) / (\mathcal{M}(A) \otimes \mathbb{K}) = \mathrm{C}^*(U\epsilon(B))$$

is the image of the natural *-homomorphism ρ from $B \rtimes_\sigma \mathbb{Z}$ into $Q^s(\mathcal{M}(A))$ defined by the covariant representation (ϵ, U). The restriction $\rho|B = \epsilon$ of ρ to B is faithful.

For $\theta \in \mathbb{C}$ with $|\theta| = 1$, the image $v(\theta) \in B$ of $(\theta 1, \theta^2 1, \dots)$ is in the center of B and satisfies $\theta\sigma(v(\theta)) = v(\theta)$. Thus, the kernel of the natural homomorphism $\rho \colon B \rtimes_\sigma \mathbb{Z} \to Q^s(\mathcal{M}(A))$ must be zero by Lemma A.26. $\square$

Let C a C*-algebra. We denote by $\overline{X}^{\mathrm{w}}$ the ultra-weak closure of a subspace X of the W^*-algebra C^{**}. Recall that $\overline{X}^{\mathrm{w}}$ is naturally isomorphic to X^{**} if X is a closed subspace of C (by Hahn-Banach extension). We write also α for the second conjugate $\alpha^{**} \in \mathrm{Aut}(A^{**})$ of $\alpha \in \mathrm{Aut}(A)$.

Lemma A.28. *Let $\alpha \in \mathrm{Aut}(A)$. There is a natural *-homomorphism η from $A^{**} \rtimes_\alpha \mathbb{Z}$ to the second conjugate $(A \rtimes_\alpha \mathbb{Z})^{**}$. η is faithful and satisfies*

(i) $\eta|A^{**}$ is the second conjugate of the inclusion map $A \hookrightarrow A \rtimes_\alpha \mathbb{Z}$, and

(ii) $\eta|(A \rtimes_\alpha \mathbb{Z})$ is the natural inclusion of $A \rtimes_\alpha \mathbb{Z}$ into it second conjugate.

Proof. A is a non-degenerate C*-subalgebra of $C := A \rtimes_\alpha \mathbb{Z}$ by Remark A.25(ii). Thus the inclusion map $\epsilon \colon A \to C$ induces a unital isometric isomorphism ϵ^{**} from A^{**} onto a W*-subalgebra of C^{**}.

α extends naturally to an automorphism of $\mathcal{M}(A) \subset A^{**}$ (that we also denoted by α). $A \rtimes_\alpha \mathbb{Z}$ is an ideal of $\mathcal{M}(A) \rtimes_\alpha \mathbb{Z}$ by Remark A.25(iii,iv). The natural inclusion of $A \rtimes_\alpha \mathbb{Z}$ into the von-Neumann subalgebra $C^{**} \subset \mathcal{L}(H)$ (for suitable H) is non-degenerate. Thus, there is a unital *-homomorphism γ from $\mathcal{M}(A) \rtimes_\alpha \mathbb{Z}$ into C^{**} that extends the natural inclusion of $A \rtimes_\alpha \mathbb{Z}$ into the von-Neumann subalgebra C^{**} of $\mathcal{L}(H)$.

Let $U_0 \in \mathcal{M}(A) \rtimes_\alpha \mathbb{Z} \subset A^{**} \rtimes_\alpha \mathbb{Z}$ the unitary that generates the copy of $\mathbb{Z}$ with $\alpha(a) = U_0 a U_0^*$, and let $U := \gamma(U_0) \in C^{**}$, then $U\epsilon(a)U^* = \epsilon(\alpha(a))$ for $a \in A$. Thus $U\epsilon^{**}(a)U^* = \epsilon^{**}(\alpha(a))$ for all $a \in A^{**}$, and there is a unique *-epimorphism η from $A^{**} \rtimes_\alpha \mathbb{Z}$ onto $\mathrm{C}^*(\epsilon^{**}(UA^{**})) = \mathrm{C}^*(\epsilon^{**}(A^{**}), U)$ with $\eta(a) = \epsilon^{**}(a)$ and $\eta(U_0) = U$ (cf. Remark A.25(v)). Clearly the restriction of η to $A \rtimes_\alpha \mathbb{Z}$ is the natural inclusion into C^{**}.

If we compose η with the normalization $C^{**} \to A^{**}\overline{\otimes}\mathcal{L}(\ell_2(\mathbb{Z}))$ of the regular representation d_{red} of $A \rtimes_\alpha \mathbb{Z}$ into $A^{**}\overline{\otimes}\mathcal{L}(\ell_2(\mathbb{Z}))$, then we get the regular representation of $A^{**} \rtimes_\alpha \mathbb{Z}$. The latter is faithful. Thus η is faithful. $\square$

Proposition A.29. *Suppose that for $\alpha \in \mathrm{Aut}(A)$, there are projections $\{e_n : \ n \in \mathbb{Z}\}$ in the center of A^{**} with*

(i) $\alpha(e_n) = e_{n+1} \leq e_n$ for $n \in \mathbb{Z}$, and

(ii) $\lim_{n\to\infty} \|a(e_{-n} - e_n)\| = \|a\|$ *for* a *in a dense subset* S *of* A.

Then every **-representation* $\rho\colon A \rtimes_\alpha \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ *with faithful restriction* $\rho|A$ *is a faithful representation of* $A \rtimes_\alpha \mathbb{Z}$.

The latter equivalently means that every non-zero closed ideal I of $A \rtimes_\alpha \mathbb{Z}$ has non-zero intersection with A.

Proof. Let $p := \bigvee_{n\in\mathbb{Z}} e_n - \bigwedge_{n\in\mathbb{Z}} e_n$. $\alpha\left(\bigwedge e_n\right) = \bigwedge e_n$ and $\alpha\left(\bigvee e_n\right) = \bigvee e_n$ in A^{**} by (i). Clearly, p is in the center of $(A)^{**}$ and $p = \sum_{n\in\mathbb{Z}} e_n - e_{n+1}$. Thus $\alpha(p) = p$ and $\epsilon\colon a \in A \to ap$ is a *-homomorphism which commutes with α. By (ii), $\|a\| = \lim_{n\to\infty} \|a(e_{-n} - e_n)\| \le \|ap\|$ for $a \in S$.

The kernel of ϵ is an α-invariant closed ideal I of A. Let $d \in I$, i.e. $dp = 0$. Since S is dense in A, there is a sequence $(b_1, b_2, \ldots)$ in S with $\lim_k \|b_k - d\| \to 0$. Then $\|b_k\| = \|b_k p\| = \|(b_k - d)p\| \le \|b_k - d\|$ implies $\lim_k \|b_k\| = 0$ and $\|d| \le \lim_k(\|b_k\| + \|b_k - d\|) = 0$. Thus $d = 0$, and ϵ is faithful.

Let $\theta \in \mathbb{C}$ with $|\theta| = 1$. Then $v(\theta) := \sum_{n\in\mathbb{Z}}(e_n - e_{n+1})\theta^n$ is a unitary in the center of $A^{**}p$ with $\theta\alpha(v(\theta)) = v(\theta)$. Hence, every *-homomorphism h from $(A^{**}p) \rtimes \mathbb{Z}$ into a C*–algebra G with faithful restriction $h|(A^{**}p)$ is itself faithful by Lemma A.26.

Let $\varrho\colon A \rtimes_\alpha \mathbb{Z} \to \mathcal{L}(H)$ a *-representation of $A \rtimes_\alpha \mathbb{Z}$ such that the restriction $\varrho|A$ is faithful, and let $\varphi\colon (A \rtimes_\alpha \mathbb{Z})^{**} \to \mathcal{L}(H)^{**}$ denote the second conjugate *-homomorphism ϱ^{**}.

Then $\varphi|A^{**}$ is also faithful, in particular $\varphi|(A^{**}p$ is faithful.

By Lemma A.28 there is a *-monomorphism η from $A^{**} \rtimes_\alpha \mathbb{Z}$ into the second conjugate of $A \rtimes_\alpha \mathbb{Z}$ such that $\eta|A^{**}$ is the second conjugate of the inclusion map $A \hookrightarrow A \rtimes_\alpha \mathbb{Z}$ and $\eta|(A \rtimes_\alpha \mathbb{Z})$ is the natural embedding from $A \rtimes_\alpha \mathbb{Z}$ in its second dual (the universal W*-crossed product $A^{**} \rtimes_{(\alpha,W^*)} \mathbb{Z}$).

Since $\alpha(p) = p$ and p is in the center of A^{**} we get that p is in the center of $A^{**} \rtimes_\alpha \mathbb{Z}$ and that $(A^{**}p) \rtimes_\alpha \mathbb{Z}$ is naturally isomorphic to $(A^{**} \rtimes_\alpha \mathbb{Z})p$. It follows that the *-homomorphism $d \mapsto dp$ from $A \rtimes_\alpha \mathbb{Z}$ into $(A^{**} \rtimes_\alpha \mathbb{Z})p$ is faithful, because $a \mapsto ap$ is faithful on A and is α-equivariant and Remark A.25(iii) applies.

Let $\lambda := \varphi \circ \eta$. Then $\lambda\colon A^{**} \rtimes_\alpha \mathbb{Z} \to \mathcal{L}(H)^{**}$ satisfies $\lambda(bp) = \varrho(b)\varphi(\eta(p))$ for $b \in A \rtimes_\alpha \mathbb{Z}$, and $\lambda|(A^{**}p)$ is faithful. Thus $\lambda|(A^{**} \rtimes_\alpha \mathbb{Z})p$ is faithful. It follows that ϱ must be faithful. □

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