Lower bounds on the radius of comparison of crossed products by minimal homeomorphisms

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A rough outline

- Introduction.
- Radius of comparison.
- Mean dimension.
- Cohomological dimension.
- Example: shifts.
- Mean cohomological independence dimension.

Introduction

Throughout, X will be a compact metric space, G will be a countable amenable group, and T will be an action of G on X, expressed as a homomorphism $g \mapsto T_g$ from G to the homeomorphisms of X.

Conjecture

Assume that T is essentially free and minimal. Then the mean dimension $\mathrm{mdim}(T)$ and the radius of comparison $\mathrm{rc}(C^*(G, X, T)$ (both described below) are related by $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

The inequality $\mathrm{rc}(C^*(G, X, T)) \leqslant \frac{1}{2}\mathrm{mdim}(T)$ is now known for some special classes of groups G. (See the talk of Zhuang Niu at this conference.) However, very little is known about the opposite inequality: we really only know about some special examples with $G = \mathbb{Z}$, the minimal subshifts of Giol and Kerr.

Introduction (continued)

Conjecture

Assume that T is essentially free and minimal. Then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

In this talk, we discuss an approach to getting lower bounds (not as good as the conjecture says, but at least nontrivial) for general (G, T), using rational cohomology.

We actually only need G to be amenable, and we don't use either minimality or any kind of freeness.

Radius of comparison 1: Cuntz comparison; d_τ

Definition

Let A be a C*-algebra. For $a, b \in M_\infty(A)_+$, we say that a is *Cuntz subequivalent to b in A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^\infty$ in $M_\infty(A)$ such that $\lim_{n\to\infty} v_n b v_n^* = a$.

It is easy to check that this restricts to Murray-von Neumann subequivalence for projections.

Definition

Let A be a unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$, the set of normalized 2-quasitraces on A. Define $d_\tau \colon M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$.

If a is a projection, then $d_\tau(a) = \tau(a)$. In general, it is the trace of the "open support of a". For example, if $f \in C(X)_+$ and τ comes from a probability measure μ on X, then

$$d_\tau(f) = \mu\big(\{x \in X \colon f(x) \neq 0\}\big).$$

Radius of comparison 2: Definition

$a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^{\infty}$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

$d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n})$, the trace of the "open support of a".

Definition

Let A be a unital C*-algebra.

1. Let $r \in [0, \infty)$. We say that A has *r-comparison* if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.
2. The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\left(\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\}\right)$$

if it exists, and ∞ otherwise.

The case $\mathrm{rc}(A) = 0$ is called *strict comparison*. Restricted to projections, this is Blackadar's Second Fundamental Comparability Question.

Radius of comparison 3: Motivation

$a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} v_n b v_n^* = a$.

$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$, the trace of the "open support of a".

$\mathrm{rc}(A)$ is the least r such that $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$ implies $a \precsim_A b$.

The quantity $\mathrm{rc}(C(X))$ is controlled by cancellation properties of vector bundles (projections) on closed subsets, and is expected to be about half the covering dimension $\dim(X)$. For "good" X, it is roughly half the covering dimension $\dim(X)$. (More below. Note: $C([0,1]^N)$ has cancellation of projections, since they are all trivial, but $\mathrm{rc}(C([0,1]^N))$ is about $N/2$.)

The case $\mathrm{rc}(A) = 0$ is called *strict comparison*, and one part of the Elliott classification conjecture says that for simple separable nuclear unital A, this condition should be equivalent to classifiability.

Mean dimension 1: Open covers; order

Throughout, X is a compact metric space, and $\mathcal{U}$ and $\mathcal{V}$ are finite collections of open subsets of X. For the definition of mean dimension they will be covers of X, but for later use we do not want to require this. Rather, we do the following.

Definition

Let X be a compact metric space, and let $Y \subset X$ be closed. An *open cover of* Y is a finite collection $\mathcal{U}$ of open subsets of X whose union contains Y.

Definition

Let X be a compact metric space, and let $\mathcal{U}$ be a finite collection of open subsets of X (not necessarily a cover). The *order* $\mathrm{ord}(\mathcal{U})$ of $\mathcal{U}$ is the least number $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n + 2$ distinct elements of $\mathcal{U}$ is empty.

For example, a cover by disjoint open sets (as is expected in a zero dimensional space) has order zero.

Mean dimension 2: Refinement; dimension

Open cover of $Y \subset X$: finite collection $\mathcal{U}$ of open subsets of X whose union contains Y.

$\operatorname{ord}(\mathcal{U})$ is the least $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n+2$ distinct elements of $\mathcal{U}$ is empty.

Definition

Let X be a compact metric space, let $Y \in X$ be closed. and let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of Y. Then $\mathcal{V}$ *refines* $\mathcal{U}$ (over Y) if every set in $\mathcal{V}$ is contained in some set in $\mathcal{U}$.

The union of the sets in $\mathcal{V}$ might be smaller than the union of the sets in $\mathcal{U}$, but must still contain Y.

Definition

Let X be a compact metric space, let $Y \in X$ be closed, and let $\mathcal{U}$ be a finite open cover of Y. Then $\mathcal{D}_Y(\mathcal{U})$ is the least possible order of any refinement of $\mathcal{U}$ over Y. Write $\mathcal{D}(\mathcal{U})$ if $Y = X$.

Mean dimension 3: Joins and images

Definition

Let X be a compact metric space, and let $\mathcal{U}$ and $\mathcal{V}$ be finite collections of open subsets of X (not necessarily covers). Then the *join* $\mathcal{U} \vee \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is

$$\mathcal{U} \vee \mathcal{V} = \{ U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V} \}.$$

If $\mathcal{U}$ covers $Y \subset X$ and $\mathcal{V}$ covers $Z \subset X$, then $\mathcal{U} \vee \mathcal{V}$ covers $Y \cap Z$.

Definition

Let X be a compact metric space, let $\mathcal{U}$ be a finite collections of open subsets of X, and let $h \colon X \to X$ be a homeomorphism. We define

$$h(\mathcal{U}) = \{ h(U) \colon U \in \mathcal{U} \}.$$

If $\mathcal{U}$ covers $Y \subset X$ then $h(\mathcal{U})$ covers $h(Y)$.

Inverse images under continuous maps are defined similarly.

Mean dimension 4: Definition

Now we consider only open covers of X.

Recall: $\mathrm{ord}(\mathcal{U})$ is the least $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n+2$ distinct elements of $\mathcal{U}$ is empty; $\mathcal{D}(\mathcal{U})$ is the least order of any refinement of $\mathcal{U}$; $\mathcal{U} \vee \mathcal{V} = \{ U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V} \}$; and $h(\mathcal{U})$ is taken setwise.

Definition

Let X be a compact metric space and let $h \colon X \to X$ be a homeomorphism. Denote by $\mathrm{Cov}(X)$ the set of finite open covers of X. Then the *mean dimension* of h is

$$\mathrm{mdim}(h) = \sup_{\mathcal{U} \in \mathrm{Cov}(X)} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}.$$

(One needs to check that the limit exists.)

In fact, mean dimension makes sense for an action T of an arbitrary countable amenable group, using Følner sets in place of intervals in $\mathbb{Z}$. We omit the details, but call it $\mathrm{mdim}(T)$ and refer to it later.

Mean dimension 5: The shift

$$\mathrm{mdim}(h) = \sup_{\mathcal{U} \in \mathrm{Cov}(X)} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}.$$

The expression $\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)$ tells you how much of the dimension of X one sees starting with $\mathcal{U}$ and applying h a total of n times. So we are looking at the "linear rate of growth of dimension with iteration of h".

The basic example is the shift on K^G for a compact metric space K, given, for $x = (x_h)_{h \in G} \in K^G$ by $T_h(x)_g = x_{h^{-1}g}$. If K is a finite complex, then one can check that $\mathrm{mdim}(T) = \dim(K)$. The simplest case is to take $\mathcal{U}$ to be the inverse image of an open cover of K under one of the projection maps $K^G \to K$. This depends on $\dim(K^n) = n \dim(K)$.

The shift is of course not free and not minimal. For example, it has fixed points.

Cohomological dimension

Definition

Let X be a compact metric space and let R be an abelian group. The *cohomological dimension of X with respect to R* is

$$\dim_R(X) = \sup\big(\{n \in \mathbb{Z}_{\geqslant 0} : \text{there is a closed subset } Y \subset X \text{ such that } \check{H}^n(X, Y; R) \neq 0\}\big).$$

To see that this isn't off by one, let $X = B^n$, the closed unit ball in $\mathbb{R}^n$, let $Y = S^{n-1} = \partial B^n$, and consider the long exact sequence

$$\cdots \longrightarrow \check{H}^{n-1}(B^n; R) \longrightarrow \check{H}^{n-1}(S^{n-1}; R) \longrightarrow \check{H}^n(B^n, S^{n-1}; R) \longrightarrow \check{H}^n(B^n; R) \longrightarrow \cdots.$$

The first and last groups shown are zero, and $\check{H}^{n-1}(S^{n-1}; R) \cong R$, so $\check{H}^n(B^n, S^{n-1}; R) \cong R$.

It is known that $\dim_R(X) \leqslant \dim_{\mathbb{Z}}(X) \leqslant \dim(X)$.

Cohomological dimension (continued)

$$\dim_R(X) = \sup\big(\big\{n \in \mathbb{Z}_{\geqslant 0} : \text{there is a closed subset } Y \subset X \text{ such that } \check{H}^n(X, Y; R) \neq 0\big\}\big).$$

It is known that $\dim_R(X) \leqslant \dim_{\mathbb{Z}}(X) \leqslant \dim(X)$. If X is a finite complex, then $\dim_R(X) = \dim(X)$ for any $R \neq 0$. If X is finite dimensional, then $\dim_{\mathbb{Z}}(X) = \dim(X)$. But $\dim_{\mathbb{Z}}(X) = 3$ and $\dim(X) = \infty$ can happen, and $\dim_{\mathbb{Q}}(X) < \dim_{\mathbb{Z}}(X)$ can happen.

Theorem (Elliott-Niu)

Let X be a compact metric space. Then

$$\mathrm{rc}(C(X)) \geqslant \begin{cases} \dfrac{\dim_{\mathbb{Q}}(X) - 1}{2} - 1 & \dim_{\mathbb{Q}}(X) \text{ is odd} \\ \dfrac{\dim_{\mathbb{Q}}(X)}{2} - 2 & \dim_{\mathbb{Q}}(X) \text{ is even} \\ \infty & \dim_{\mathbb{Q}}(X) = \infty. \end{cases}$$

Cohomological dimension (continued)

Theorem (Elliott-Niu; roughly stated)

Let X be a compact metric space. Then $\mathrm{rc}(C(X))$ is at least about $\frac{1}{2}\dim_{\mathbb{Q}}(X) - 2$.

For any compact metric X, it is known that $\mathrm{rc}(C(X)) \leqslant \frac{1}{2}(\dim(X) - 1)$ (unless the right hand side is negative, in which case $\mathrm{rc}(C(X)) = 0$).

In particular, for finite complexes, $\dim(X) = \dim_{\mathbb{Q}}(X)$, so Elliott and Niu give lower bounds for $\mathrm{rc}(C(X))$ in terms of $\dim(X)$. For general X, there are no known nontrivial lower bounds for $\mathrm{rc}(C(X))$ in terns of $\dim(X)$.

Heuristically, the idea of the proof is to construct complex vector bundles over a suitable closed subset $Y \subset X$ with bad comparison properties, interpret them as projections in $M_n(C(Y))$ for sufficiently large n, and extend to positive elements in $M_n(C(X))$. The choice of $\dim_{\mathbb{Q}}(X)$ rather than $\dim_{\mathbb{Z}}(X)$ (which is sometimes bigger) is dictated by the fact that the Chern character takes values in $\check{H}^*(X;\mathbb{Q})$, not in $\check{H}^*(X;\mathbb{Z})$. One can't use K-theory instead, because it is only $\mathbb{Z}/2\mathbb{Z}$-graded.

The shift on S^k for k even

Let $k \in 2\mathbb{Z}$, and let T be the shift action of G on $(S^k)^G$.

For the main lemma, we consider a general action.

Lemma

Let T be an action of G on a connected compact metric space X. Let $\alpha \colon G \to \mathrm{Aut}(C(X))$ be the corresponding action on $C(X)$. Set $A = C^*(G, X, T)$. Let $n \in \mathbb{Z}_{>0}$. Let $p, q \in M_n(C(X))$ be projections, suppose that q is a constant projection, and suppose that $p \precsim_A q$. Then for every $\varepsilon > 0$ there are nonempty finite subsets $P, Q \subset G$ such that $\mathrm{card}(Q) < (1 + \varepsilon)\mathrm{card}(P)$ and

$$\bigoplus_{g \in P} (\mathrm{id}_{M_n} \otimes \alpha_g^{-1})(p) \precsim_{C(X)} \bigoplus_{g \in Q} q.$$

Sketch of proof of the lemma

We are assuming $p \precsim_A q$, and we want to relate p and q over $C(X)$.

Sketch of proof.

Choose $v \in M_n(A)$ such that $\|v^* q v - p\| < \frac{1}{4}$. (Could get $v^* q v = p$, but not in the version for positive elements.) We can assume v is in the $n \times n$ matrices over the algebraic crossed product, that is, with u_g being the canonical unitary in $C^*(G, X, T)$,

$$v = \sum_{g \in G_0} v_g (1_{M_n} \otimes u_g)$$

with $G_0 \subset G$ finite and $v_g \in M_n(C(X))$ for $g \in G_0$. We can assume G_0 is symmetric and contains 1.

Choose $\delta > 0$ suitably small. (It depends on v.) Choose a sufficiently good G_0-Følner set $S \subset G$, and choose a sufficiently good S-Følner set $F \subset G$. We can assume S and F are symmetric and contain 1. Set $P = \bigcap_{h \in S} hF$ and $Q = SF$. With appropriate choices, we get $\text{card}(Q) < (1 + \varepsilon)\text{card}(P)$.

Sketch of proof of the lemma (continued)

Sketch of proof (continued).

We chose a good G_0-Følner set $S \subset G$, and a good S-Følner set $F \subset G$. We defined $P = \bigcap_{h \in S} hF$ and $Q = SF$.

Define $\Delta = \text{card}(S)^{-1} \chi_S * \chi_F$. This function has the properties:

1. $0 \leqslant \Delta \leqslant 1$.
2. $\Delta(g) = 1$ for $g \in P$.
3. $\Delta(g) = 0$ for $g \in G \setminus Q$.
4. $|\Delta(t^{-1}g) - \Delta(g)| < \delta$ for all $t \in G_0$ and $g \in G$.

Sketch of proof of the lemma (continued)

Sketch of proof (continued).

Represent $A = C^*(G, X, T)$ on the Hilbert $C(X)$-module $l^2(G, C(X))$ in the standard way: for $b \in C(X)$, $\xi \colon G \to C(X)$ in $l^2(G, C(X))$, and $g, h \in G$,

$$(\pi(b)\xi)(g) = \alpha_g^{-1}(b) \cdot \xi(g) \qquad \text{and} \qquad (\pi(u_h)\xi)(g) = \xi(h^{-1}g).$$

Use the same letter for the corresponding representation of $M_n(A)$ on $l^2(G, C(X))^n$. Further define multiplication operators on $l^2(G, C(X))$ by

$$d_0 = m(\Delta), \qquad e_0 = m(\chi_P), \qquad \text{and} \qquad f_0 = m(\chi_Q),$$

and set

$$d = \mathrm{id}_{M_n} \otimes d_0, \qquad e = \mathrm{id}_{M_n} \otimes e_0, \qquad \text{and} \qquad f = \mathrm{id}_{M_n} \otimes f_0.$$

Sketch of proof of the lemma (continued)

Sketch of proof (continued).

Recall:

1. $0 \leqslant \Delta \leqslant 1$.
2. $\Delta(g) = 1$ for $g \in P$.
3. $\Delta(g) = 0$ for $g \in G \setminus Q$.
4. $|\Delta(t^{-1}g) - \Delta(g)| < \delta$ for all $t \in G_0$ and $g \in G$.

We have $fd = d$ and $de = e$. Also, $f\pi(M_n(A))f \subset L\big(l^2(Q, C(X))^n\big)$, which is a matrix algebra of size $n\text{card}(Q)$ over $C(X)$. Furthermore, $\|d_0\pi(u_g) - \pi(u_g)d_0\| < \delta$ for $g \in G$, so (assuming δ is small enough)

$$\|d\pi(v) - \pi(v)d\| \leqslant \delta \sum_{g \in G_0} \|v_g\| < \frac{1}{2\|v\|}.$$

Sketch of proof of the lemma (continued)

Sketch of proof (continued).

Recall that $\| v^* q v - p \| < \frac{1}{4}$.

Set $w = d^{1/2} \pi(v) d^{1/2}$. Then one can check that

$$\left\| w^* d^{1/2} \pi(q) d^{1/2} w - d^{3/2} \pi(p) d^{3/2} \right\| < 1.$$

So, using $ed = e$ and $fd = d$,

$$\left\| e w^* d^{1/2} f \pi(q) f d^{1/2} w e - e \pi(p) e \right\| < 1.$$

Now $e \pi(p) e$, taken in $L(^2(Q, C(X))^n)$, is $0 \oplus \bigoplus_{g \in P} (\mathrm{id}_{M_n} \otimes \alpha_g^{-1})(p)$. Also, the inequality implies that $p \pi(e) p$ is Murray-von-Neumann subequivalent to $f \pi(q) f$, which is $\bigoplus_{g \in Q} q$. □

Back to the shift

Back to the shift on $(S^k)^G$. For a complex vector bundle E over a connected space X, recall Chern classes $c_j(E) \in \check{H}^{2j}(X; \mathbb{Z})$ and the total Chern class $c(E) = 1 + c_1(E) + \cdots + c_r(E)$, with $r = \operatorname{rank}(E)$. Further recall the sum formula (cup product on the right): $c(E \oplus F) = c(E)c(F)$, and that if F is trivial then $c(F) = 1$. Let ε be a generator of $\check{H}^k(S^k; \mathbb{Z})$. There is a vector bundle E_0 over S^k with (complex) rank $k/2$ such that $c(E_0) = 1 + m\varepsilon$ with $m \neq 0$.

Over $(S^k)^n$ take the direct sum E of the pullbacks of E_0 by the projections to the factors. The ring $\check{H}^*((S^k)^n; \mathbb{Z})$ is commutative and generated by the corresponding pullbacks $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of ε, with the relation $\varepsilon_l^2 = 0$ for all l. So $c(E) = \prod_{l=1}^n (1 + m\varepsilon_l)$. If $E \oplus F$ is trivial, then $c(F) \prod_{l=1}^n (1 + m\varepsilon_l) = 1$, which implies that $c(F) = \prod_{l=1}^n (1 - m\varepsilon_l)$. Since this has a nonzero term $(-m)^n \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n$ in degree kn, it follows that $\operatorname{rank}(F) \geqslant nk/2$.

Radius of comparison for the shift

In the lemma, take p to be the projection on the pullback of E_0 by the projection map to one of the factors. Take q to be a trivial projection. The lemma gives P and Q such that $\mathrm{card}(Q) < (1+\varepsilon)\mathrm{card}(P)$ and

$$\bigoplus_{g \in P} (\mathrm{id}_{M_n} \otimes \alpha_g^{-1})(p) \precsim_{C(X)} \bigoplus_{g \in Q} q.$$

So, by the previous slide at the first step,

$$k\mathrm{card}(P) \leqslant \mathrm{rank}(q)\mathrm{card}(Q) < (1+\varepsilon)\mathrm{rank}(q)\mathrm{card}(P).$$

Since $\varepsilon > 0$ is arbitrary, we get $\mathrm{rank}(q) \geqslant k$. That is, if we take a trivial projection q of rank less than k, the rank $k/2$ projection p is not Murray-von-Neumann subequivalent to q in the crossed product.

This doesn't quite show that $\mathrm{rc}(C^*(G, X, T)) \geqslant k/2$, but with a bit more work one does in fact get that fact.

Mean cohomological independence dimension

Definition

Let G be a countable abelian group, let X be a compact metric space, and let T be an action of G on X. Let R be a unital commutative ring. Let $k \in 2\mathbb{Z}$. For $d \in \mathbb{R}$ we say that $\mathrm{mcid}_k(T; R) \geqslant d$ if the following happens.

For every $\varepsilon > 0$ there are a closed subset $Y \subset X$, a finite collection $\mathcal{U}$ of open subsets of X which covers Y, such that $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$, and $\eta \in \check{H}^k(Y; \mathcal{U}; R)$ (Čech classes using $\mathcal{U}$) such that for every finite subset $G_0 \subset G$ and every $\delta > 0$ there are a (G_0, δ)-invariant nonempty finite set $F \subset G$ and a subset $F_0 \subset F$ for which the following happen:

1. The cup product of $T_g^*(\eta)$, over all $g \in F_0$, which makes sense as an element of $\check{H}^{k \cdot \mathrm{card}(F_0)}\left(\bigcap_{g \in F_0} T_g^{-1}(Y); R\right)$, is nonzero.
2. $\dfrac{k \cdot \mathrm{card}(F_0)}{\mathrm{card}(F)} > d - \varepsilon$.

Why subsets of X and the Følner set?

We will take $R = \mathbb{Q}$. The condition:

For every $\varepsilon > 0$ there are a closed subset $Y \subset X$, an open cover $\mathcal{U}$ of Y, with $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$, and $\eta \in \check{H}^k(Y; \mathcal{U}; \mathbb{Q})$ such that for every finite $G_0 \subset G$ and every $\delta > 0$ there are a (G_0, δ)-invariant nonempty finite $F \subset G$ and $F_0 \subset F$ for which:

1. The cup product of $T_g^*(\eta)$, over all $g \in F_0$, is nonzero.
2. $\dfrac{k \cdot \mathrm{card}(F_0)}{\mathrm{card}(F)} > d - \varepsilon$.

Recall that our collections of open sets need not cover X. Then $\mathcal{D}_Y(\mathcal{U})$ is the least order of a refinement of $\mathcal{U}$ over Y. We need nontrivial cohomology, so, as in the definition of cohmological dimension, we need to use closed subsets $Y \subset X$. (For example, maybe $X = [0, 1]^G$.)

Unlike for the full shift, we can't expect the cup product of $T_g^*(\eta)$, over all $g \in F$, to be nonzero. We can only require this to happen for a subset $F_0 \subset F$, preferably of large "density". The left hand side in (2) is density times dimension in cohomology. (This causes unexpected trouble later.)

Why rational coefficients?

For every $\varepsilon > 0$ there are a closed subset $Y \subset X$, an open cover $\mathcal{U}$ of Y, with $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$, and $\eta \in \check{H}^k(Y; \mathcal{U}; \mathbb{Q})$ such that for every finite $G_0 \subset G$ and every $\delta > 0$ there are a (G_0, δ)-invariant nonempty finite $F \subset G$ and $F_0 \subset F$ for which:

1. The cup product of $T_g^*(\eta)$, over all $g \in F_0$, is nonzero.
2. $\dfrac{k \cdot \mathrm{card}(F_0)}{\mathrm{card}(F)} > d - \varepsilon$.

We need rational coefficients (just like in Elliott-Niu) to get from a cohomology class to a vector bundle (plus technical details). The Chern character is an isomorphism from rational K-theory to rational cohomology. To make it additive instead of multiplicative, one needs complicated rational polynomials in the Chern classes.

We need our vector bundle to have no Chern classes in degree greater than k. (The calculation above with $1 + m\varepsilon$ is spoiled if there are higher degree terms.) Having $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$ does this: all higher even cohomology from this cover is zero.

The theorem

Condition for $\operatorname{mcid}_k(T;\mathbb{Q}) \geqslant d$: for every $\varepsilon > 0$ there are a closed subset $Y \subset X$, an open cover $\mathcal{U}$ of Y, with $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$, and $\eta \in \check{H}^k(Y;\mathcal{U};\mathbb{Q})$ such that for every finite $G_0 \subset G$ and every $\delta > 0$ there are a (G_0, δ)-invariant nonempty finite $F \subset G$ and $F_0 \subset F$ for which:

1. The cup product of $T_g^*(\eta)$, over all $g \in F_0$, is nonzero.
2. $k \cdot \text{card}(F_0)/\text{card}(F) > d - \varepsilon$.

Theorem

Let G be a countable abelian group, let X be a compact metric space, and let T be an action of G on X. Then for any $k \in 2\mathbb{Z}_{\geqslant 0}$, we have $\text{rc}(C^*(G, X, T)) \geqslant \operatorname{mcid}_k(T;\mathbb{Q}) - \frac{k}{2}$.

We want $\text{rc}(C^*(G, X, T)) \geqslant \frac{1}{2}\operatorname{mcid}_k(T;\mathbb{Q})$. If the "density" $\text{card}(F_0)/\text{card}(F)$ is 1, then $\operatorname{mcid}_k(T;\mathbb{Q}) = k$, and that is what we get. In general, $\operatorname{mcid}_k(T;\mathbb{Q}) < k$. The trouble is the effect of the terms $T_g^*(\eta)$ for $g \in F \setminus F_0$. These should contribute nothing, but maybe they actually interfere.

Final remarks

Theorem

Let G be a countable abelian group, let X be a compact metric space, and let T be an action of G on X. Then for any $k \in 2\mathbb{Z}_{\geqslant 0}$, we have $\mathrm{rc}(C^*(G, X, T)) \geqslant \mathrm{mcid}_k(T; \mathbb{Q}) - \frac{k}{2}$.

Remark

We always have $\mathrm{mcid}_k(T; \mathbb{Q}) = \mathrm{mdim}(T)$, and in good cases it is at least nearly equal.

There are now known examples of "thick" minimal subshifts for arbitrary amenable groups G, to which to apply this. (Due to Dou.)