Recall: A Banach algebra is a $\mathbb{C}$-Banach space A, also an algebra over $\mathbb{C}$, with $\|ab\| \le \|a\|\,\|b\|$. If unital: $\|1\| = 1$.

Def A Banach $*$-alg. is Ban. alg with $a \mapsto a^*$ s.t.

(1) Conj linear

(2) $(ab)^* = b^* a^*$ — It follows that $1^* = 1$ if unital.

(3) $(a^*)^* = a$.

(4) $\|a^*\| = \|a\|$.

Def A above is a **C^*-alg** if also $\|a^* a\| = \|a\|^2$ $\forall a \in A$ ($\|a^* a\| \le \|a\|^2$ is automatic).

Looks innocuous, but very powerful, very restrictive.

Ex (1) $M_n = M_n(\mathbb{C}) = L\left(\ell^2(\{1, \dots, n\})\right)$ with operator norm, a^* is the conjugate transpose of a.

(2) If X is cpt Hausdorff, then $C(X)$, with $\|\ \|_\infty$, and $f^*(x) = \overline{f(x)}$, is a commutative unital C^*-alg.

(3) H Hilbert space, then $L(H)$, with operator norm, a^* given by $\langle a^* \xi, \eta \rangle = \langle \xi, a\eta \rangle$ $\forall \xi, \eta \in H$, is a C^*-alg. (Ex 1 is a special case).

(4) Any norm closed self-adj subalg $A \subset L(H)$ is a C^*-alg. ($a \in A \Rightarrow a^* \in A$)

Thm Every C^*-alg is isomorphic to one as in example (4).

Pf: lots of work; see standard courses.

Thm Every commutative unital C^*-alg is isom. to an alg as in Ex. (2). (with cont. maps)

More is true: $X \mapsto C(X)$ is a contravariant category eq. from cpt T_2 spaces to commutative unital C^*-algs with unital homomorphisms. ($*$-hom: $\varphi(a)^* = \varphi(a^*)$)

It means: also $X \cong Y \iff C(X) \cong C(Y)$. (↑ !!)

If $h: X \to Y$ is cont, get a hom $\varphi: C(Y) \to C(X)$ by $\varphi(f) = f \circ h$.

Moreover, given $\psi: C(Y) \to C(X)$, it comes from some $h: X \to Y$, in this way.

Also needs work; basic idea: if have $C(X)$, recover X as the max ideal space of $C(X)$.

This leads to idea: noncomm C^*-algs are "noncommutative cpt spaces".

This has led to a lot of research, including applications to other areas.

$\subset C^*$-algs.

Defn $\varphi: A \to B$ is a hom if φ is an alg hom. with $\varphi(a^*) = \varphi(a)^* \ \forall a \in A$.

Striking result easily via subject:

(1) Automatically have $\|\varphi(a)\| \leq \|a\| \ \forall a \in A$.

(2) φ has closed range. (3) If φ is inj-tive, then $\|\varphi(a)\| = \|a\| \ \forall a \in A$.

Note: norm closed ideals are automatically selfadj, and quotient is a C^*-alg. (needs work too).

Nice examples: (1) Upper triangular matrices (not selfadj).

(2) Disk algebra $A(D)$ fcns cont on closed unit disk $\overline{D}$, hol. on D.

If has an adjoint: $f^*(z) = \overline{f(z)}$, very badly behaved.

(3) $L^1(G)$ for loc cpt gp G, Specialize to discrete G: $\ell^1(G)$. $a^*(g) = \overline{a(g^{-1})}$, mult. is convolution, $\|\ \|_1$. Somewhat close to being a C^*-alg: put on a new norm, and complete. The result is $C^*(G)$, the C^*-alg of G.

If G is abelian, get $C(\hat{G})$, $\hat{G}$ dual gp. Ex: $G = \mathbb{Z}$. Then $\ell^1(\mathbb{Z})$ "is" the fcns on S^1 with abs. convergent Fourier series, and $C^*(G)$ is $C(S^1)$.

(4) $L(\ell^p(\{1, \dots, n\}))$ with $p \in [1, \infty] \setminus \{2\}$. This is M_n; adjoint is cont, but norm is wrong. (Conjugate transpose.)

In (3), don't get $\|a^*a\| = \|a\|^2$ (C^*-condition). Note: If $G \neq \{1\}$, never get this. (but OK for $G = \mathbb{Z}/2\mathbb{Z}$.)

Compute norm in $L(\ell^2(\{1, \dots, n\}))$: find largest eigenvalue of a^*a, and take $\sqrt{\ }$.

$p = 1$ or $p = \infty$: Take 1-norms of rows or columns, and take max.

$p = \infty$ ↑ ↕ ↑ $p = 1$

$p = 1$ $p = \infty$

As far as I know, for all other p numerically really bad.

Defns of selfadjoint ($a^* = a$), normal ($a^*a = aa^*$), unitary ($a^*a = aa^* = 1$) generalize to C^*-algs.

Thm If $a^* = a$ then $\mathrm{sp}(a) \subset \mathbb{R}$ $\quad \{\lambda \in \mathbb{C} : \lambda - a$ not invertible$\}$

Say $a \geq 0$ if also $\mathrm{sp}(a) \subset [0, \infty)$.

Thm (no pf): Let $a \in A$. Then TFAE: (1) $a^* = a$ and $\mathrm{sp}(a) \subset [0, \infty)$.

(2) $\exists\, b$ st $a = b^*b$. (3) $\exists$ selfadj b st. $a = b^2$.

In $L(H)$, eq. to: $\langle a\xi, \xi\rangle \geq 0 \ \forall \xi \in H$.
Can go badly wrong without C^* condition. In $A(\overline{D})$, the fcn $f(\zeta) = \zeta$ is selfadj but $sp(f) = \overline{D}$. Worse, $f^2 = f^* f$, is $\zeta \mapsto \zeta^2$. Its spectrum is $\overline{D}$.

If a, b are selfadj, write $a \leq b$ if $b - a \geq 0$. This gives an order.
If also $c^* = c$, then $a \leq b \Rightarrow a + c \leq b + c$. Also true: $x \in A$, $a \leq b \Rightarrow x^* a x \leq x^* b x$. (pf: If $b - a = c^* c$ then $x^* b x - x^* a x = (cx)^* (cx)$).
Almost everything else is false. Ex: $0 \leq a \leq b \not\Rightarrow a^2 \leq b^2$. Mostly, if pf is not easy, it is not true.
True in $C(X)$, but only there.
In $C(X)$, get good behavior not seen elsewhere: $f \geq 0$ iff $f(x) \geq 0 \ \forall x$.
f is selfadj iff $f(x)$ is real $\forall x \in X$. $f \leq g$ iff $f(x) \leq g(x) \ \forall x \in X$.

Functional calculus: Suppose A unital, $a \in A$ is normal. Then $C^*(a, 1)$ is commutative unital, so $\cong C(X)$ for some X. It turns out that X "$=$" $sp(a)$. Thus get $C(sp(a)) \to C^*(a, 1) \hookrightarrow A$, a hom, write it as $f \longmapsto f(a)$.
Fact: Unique in a suitable sense, and extends hol. fcnl calculus. In particular, if f is hol. on a nbhd of $sp(a)$, to compute $f(a)$, only need values of f on $sp(a)$.
If $a^* = a$ and in $L(H)$, by spectral thm, there is a proj valued measure E st. $a = \int_{sp(a)} \lambda \, dE(\lambda)$. Then $f(a) = \int_{sp(a)} f(\lambda) \, dE(\lambda)$.

Several more examples of C^*-algs
(5) $C(X, M_n)$ with pointwise mult, $\|f\| = \sup_X \|f(x)\|$, $f^*(x) = f(x)^*$.
↑ could use any C^*-alg. here.
(6) $K(\ell^2)$, $Q = L(\ell^2)/K(\ell^2)$.
(7) Irrational rotation algebras. Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $u, v \in L(\ell^2(\mathbb{Z}))$ be the unitaries $(u\xi)(n) = \xi(n-1)$ and $(v\xi)(n) = e^{2\pi i n\theta} \xi(n)$. One can check that $vu = e^{2\pi i\theta} uv$.
Take $A_\theta = C^*(u, v) \subset L(\ell^2(\mathbb{Z}))$.
$((vu)\xi)(n) = e^{2\pi i n\theta} (u\xi)(n) = e^{2\pi i n\theta} \xi(n-1) = e^{2\pi i\theta}$.
$((uv)\xi)(n) = (v\xi)(n-1) = \boxed{e^{2\pi i(n-1)\theta} \xi(n-1)}$

Thm (1) A_θ is simple (no nontrivial closed 2-sided ideals)

(2) One gets the same C^*-alg. as $C^*(u_0, v_0)$ for any two unitaries on any Hilbert space with $v_0 u_0 = e^{2\pi i\theta} u_0 v_0$ (with $u \mapsto u_0$, $v \mapsto v_0$).

(3) If $\theta_1, \theta_2 \in (0, \frac{1}{2}) \setminus \mathbb{Q}$ are distinct, then $A_{\theta_1} \ncong A_{\theta_2}$. All taken with proved ~ 1980.

Crossed products. Ways of making new C^*-algs from old ones:

crossed products, direct limits, tensor products (various kinds of free products), only talk about these, under conditions: groups is discrete, usually amenable.

Def A C^*-alg, G gp (with discrete top). An action α of G on A is hom $G \to Aut(A)$, written $g \mapsto \alpha_g$ (so get $\alpha_g(a)$ for $g \in G$, $a \in A$).

(If G not discrete, cont. conditions)

Note: An action of $\mathbb{Z}$ is "the same as" an automor of A. If $\alpha \in Aut(A)$, $n \mapsto \alpha^n$ is an action of $\mathbb{Z}$.

Also: If X loc cpt, then actions on $C_0(X)$ are "the same as" cont. actions of G on X. Given $(g, x \mapsto gx)$, define $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C_0(X)$, $x \in X$.

Exs (mostly on $C(X)$).

(1) Trivial action of G on $\mathbb{C}$ (X is a point).

(2) G acts by translation on G, so on $C_0(G)$. (gx is the group mult.)

(3) $\mathbb{Z}/n\mathbb{Z}$ acts on S^1 by rotation by $2\pi/n$.

(4) $\mathbb{Z}$ acts on S^1 by rotation by $2\pi\theta$ (interesting case is $\theta \notin \mathbb{Q}$).

(5) If $u \in A$ unitary, then $a \mapsto uau^*$ is an auto, so gives action of $\mathbb{Z}$.

(6) S^1 acts A_θ by $\alpha_\zeta(u) = \zeta u$, $\alpha_\zeta(v) = \zeta v$. (Main point: $(\zeta v)(\zeta u) = e^{2\pi i\theta}(\zeta u)(\zeta v)$.)
($vu = e^{2\pi i\theta} uv$)

The way to see $A_{\theta_1} \ncong A_{\theta_2}$ is to compare their ordered K_0-groups $\mathbb{Z} + \theta_1\mathbb{Z}, \mathbb{Z}, \mathbb{Z} + \theta_2\mathbb{Z} \subset \mathbb{R}$

Given $\alpha: G \to Aut(A)$, the crossed product $C^*(G, A, \alpha)$ is the universal C^*-alg gen. by
↑ G discrete, A unital
a copy of A and a unitary repn $g \mapsto u_g$ of G such that $u_g a u_g^* = \alpha_g(a)$ ↙ $u_g^* = u_{g^{-1}}$, $u_{gh} = u_g u_h$.
$\forall g \in G$, $a \in A$.

One way to construct it, if G is also amenable. Exs: Finite gps are amenable, $\mathbb{Z}$ is amenable. All solvable gps, including all abelian gps, are amenable. F_2 (free gp on 2 generators) and $SL_2(\mathbb{Z})$ are not amenable)

Choose any faithful unital repn $\pi_0 : A \to \mathcal{L}(H_0)$. Set $H = \ell^2(G, H_0)$, set of seqs in H_0, $(\xi(g))_{g\in G}$ st. $\sum_{g\in G} \|\xi(g)\|^2 < \infty$. Define repn $g \mapsto u_g \in \mathcal{L}(H)$ by $(u_g\xi)(h) = \xi(g^{-1}h)$, and repn $\pi : A \to \mathcal{L}(H)$ by $(\pi(a)\xi)(h) = \pi_0(\alpha_h^{-1}(a))(\xi(h))$. Check: $u_g \pi(a) u_g^* = \pi(\alpha_g(a))$. "Define" $C^*(G, A, \alpha)$ to be the C^*-alg gen by $\{u_g : g\in G\}$ and $\pi(A)$.

Fact. Does not depend on choice of π_0. (needs work).

What does a crossed product look like?

(1) For trivial action on G, get $C^*(G)$ as mentioned before (the "largest" C^* completion of $\ell^1(G)$.) ← G finite

(2) Let G act on G by translation, and let $\alpha : G \to C(G)$ be corresponding action.

Claim: $C^*(G, C(G), \alpha) \cong M_n$ with $n = \mathrm{card}(G)$. Write down a formula for a system of matrix units indexed by G rather than by $\{1, \dots, n\}$. Let $\delta_g \in C(G)$ be $\mathbb{1}_{\{g\}}$.

Set $e_{g,h} = \delta_g u_{gh^{-1}}$. Check that (exercise!) $e_{g_1,h_1} e_{g_2,h_2} = \begin{cases} e_{g_1,h_2} & h_1 = g_2 \\ 0 & h_1 \neq g_2, \end{cases}$

and $(e_{g,h})^* = e_{h,g}$ Will need $u_g \delta_h u_g^* = \alpha_g(\delta_h) = \delta_{gh}$.

(3) $\mathbb{Z}/n\mathbb{Z} = G$ acting on S^1 by rotation by $2\pi/n$. This action has orbits: for $z \in S^1$, get $Gz = \{z, e^{2\pi i/n} z, e^{4\pi i/n} z, \dots, e^{2(n-1)\pi i/n} z\}$. Here, the orbit is a copy of G, and they are parametrized by the "orbit space" S^1/G (space of orbits) another copy of S^1, thought of as $[0, 2\pi/n]$ with endpts identified, rather than $[0, 2\pi]$ with endpts identified. It turns out that one gets a (possibly) twisted version of $C(S^1, M_n)$, think of as $C(S^1/G, C^*(G, C(G)))$. In fact, one gets exactly $C(S^1, M_n)$. (The "twisting" is trivial.)

orbit space. ↑ from above.

(4). Irrational rotations: by $e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. $G = \mathbb{Z}$

Orbits: Gz, for $z \in S^1$, $Gz = \{e^{2\pi i n\theta} z : n \in \mathbb{Z}\}$. It is dense in S^1.

(differently: S^1/G is an uncountable set with the indiscrete topology).

They can't be separated like in example (3). The action is free: $gz = z \Rightarrow g = 1$. ← $G = \mathbb{Z}$

The resulting C^*-alg. is simple, and (exercise, given things on A_θ) $e^{2\pi i n\theta} z = z \Rightarrow n = 0$.

is seen to be A_θ, with u above being u_1, v above being $v(z) = z$ in $C(S^1)$.

Def Action of $G \curvearrowright X$ is **free** if $gx = x$ for some $x \Rightarrow g = 1$.
(every nontrivial gp elt moves every elt of X).

minimal if Gx is dense $\forall x \in X$,
(no nontrivial closed G-invariant subsets, eq: no nontrivial open inv. subsets).
nontrivial: $\emptyset, X$ — discrete.

Thm G amenable, X cpt, action free and minimal. Then $C^*(G,X)$ is simple ($\leftarrow C^*(G, C(X))$.
(also, stably finite and has tracial states).

In particular, take $G = \mathbb{Z}$. If $h: X \to X$ generates the action, and X is infinite and h is minimal, then h is free.

The action of G on G by translation is free and minimal; in fact, no nontrivial invariant subsets, closed or not. ($\leftarrow$ loc cpt)

If G not cpt, then is not an action on a cpt space. Crossed product is $K(L^2(G))$.

Full Thm Let G be discrete (maybe need sfble), X cpt (maybe need metrizable).
Assume the action is "essentially free". Then $C^*_r(G,X)$ is simple.

Without amenability, the construction of $C^*(G,A,\alpha)$ above actually gives $C^*_r(G,A,\alpha)$

Thm G amenable (discreteness not needed). Then $C^*(G,A,\alpha) \to C^*_r(G,A,\alpha)$ is an isom.

Thm If $C^*(G) \to C^*_r(G)$ is an isom, then G is amenable. (for loc cpt gps).

Suppose G acts on $C(X)$. For $g \in G$ get $\alpha_g \in \mathrm{Aut}(C(X))$. Then there is a cont map $T_g: X \to X$ such that $\alpha_g(f) = f \circ T_g$ $\forall f \in C(X)$.
(Comes from: unital hom $C(Y) \to C(X)$ is of form $f \mapsto f \circ h$ for some cont $h: X \to Y$)
h is unique.

By uniqueness, $T_g \circ T_h = T_{hg}$, and $T_1 = \mathrm{id}_X$. So T_g is a homeo $\forall g$, and thus $(g,x) \mapsto T_g^{-1}(x)$ is an action of G on X.