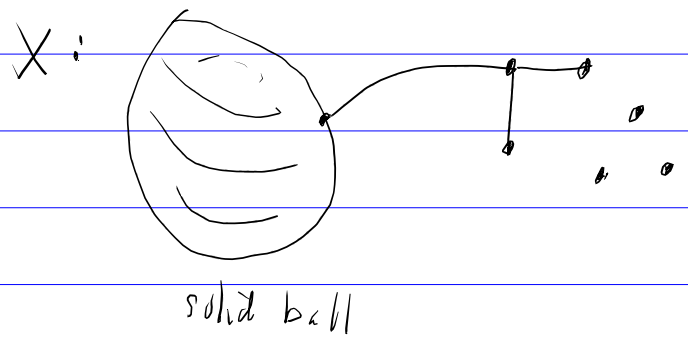
Dimension and mean dimension, $\dim(X)$.

↖ covering dimension, for cpt metric spaces. (others, like large ind, inductive dim, agree on cpt metric spaces, but not in general) (Hausdorff dim is quite different).

$\dim(X) \in \mathbb{Z}_{\geq 0}$ ($\dim(\emptyset) = -1$, however).

X:



solid ball

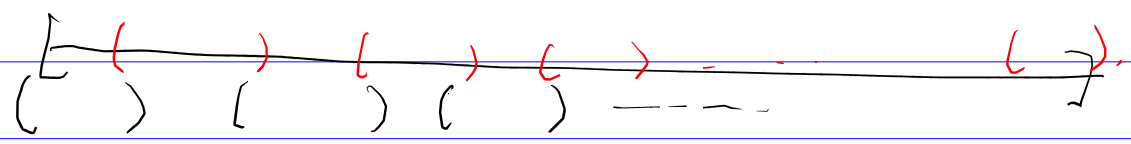
$\dim(X) = 3$, almost "largest part"

One still has some bad behavior, even for cpt metric spaces:

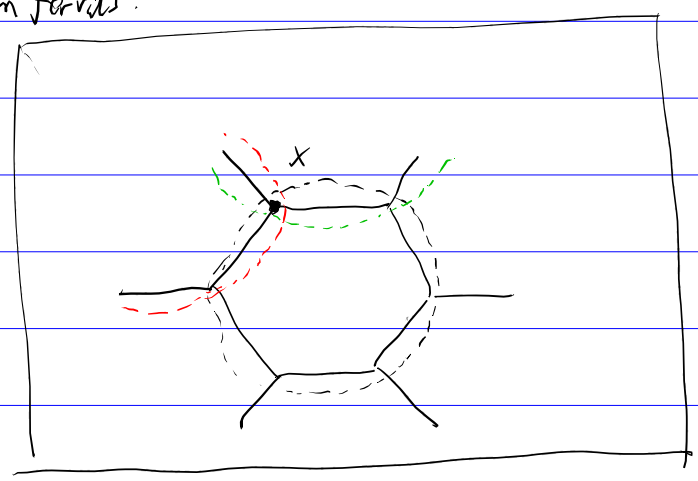
$\dim(X \times Y) \leq \dim(X) + \dim(Y)$, but "$<$" can happen, even if $X = Y$.

Start with Cantor set X. It has "arb. fine" covers by finitely many disjoint open sets.

$X = [0,1]$ Cover by such open sets which overlap as little as possible.



No two black intervals intersect, no two red intervals intersect. So each pt is in at most two intervals.



x is in three sets simultaneously, but not four.

Defn $\mathrm{Cov}(X)$ is the set of finite open covers of X.

Defn For $\mathcal{U} \in \mathrm{Cov}(X)$, define $\mathrm{ord}(\mathcal{U})$ to be the largest n s.t. $\exists\, U_1, \ldots, U_n \in \mathcal{U}$ distinct with $\bigcap U_k \neq \emptyset$, and subtract 1.

So a cover by disj open sets has order 0.

Defn $\mathcal{U} \prec \mathcal{V}$ ($\mathcal{U}$ refines $\mathcal{V}$) if every set in $\mathcal{U}$ is contained in a set in $\mathcal{V}$.

Defn $D(\mathcal{U})$ is the least order of a refinement of $\mathcal{U}$.

Defn $\dim(X) = \sup_{\mathcal{U} \in \mathrm{Cov}(X)} D(\mathcal{U})$.

Thm $\dim([0,1]^d) = d$. Exercise: Prove for $d = 0, 1$. Prove $\leq$ for $d = 2$.

Hard direction is $\geq$. It needs Brouwer fixed pt thm or similar alg. topology.
More generally: if X is a d-dimensional manifold (possibly with boundary), or a d-dim simplicial or finite CW complex, in conventional sense of dim, then $\dim(X) = d$.
Fact: $Y \subset X$ closed $\Rightarrow \dim(Y) \leq \dim(X)$.

General situation: X cpt metric, $h: X \to X$ a homeo. For purposes here, basic ex is the shift on $X = K^{\mathbb{Z}}$ for for K cpt metric. $h_K(x)_n = x_{n+1}$. (backwards shift)
x seq

$x = (\ldots\ldots x_{-2}, x_{-1}, x_0, x_1, \ldots\ldots)$ (Notation with shifts is confusing).

$h_K(x) = (\ldots\ldots x_{-1}, x_0, x_1, x_2 \ldots)$

(If G is discrete, there is G-shift on K^G, similar defn.)
$\mathrm{mdim}(h)$, the mean dim. of h, tells you "how many more "dimensions" of X do you see with each iteration of h". Idea: Gromov. Defn in paper of Lindenstrauss-Weiss. Want: $\mathrm{mdim}(h_K) = \dim(K)$ for reasonable K. (need $\dim(K^n) = n\dim(K)$; for bad K can have "$<$")
↖ Finite CW cxs, simplicial complexes, also cpt mfds.

Defn $\mathcal{U} \in \mathrm{Cov}(X)$. Then $h(\mathcal{U}) = \{h(U) : U \in \mathcal{U}\}$.

Defn $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$. Then $\mathcal{U} \vee \mathcal{V}$ (join) $= \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ (Least common refinement)

Defn $\mathrm{mdim}(h) = \sup_{\mathcal{U} \in \mathrm{Cov}(X)} \lim_{n\to\infty} \frac{D(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee h^{-2}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U}))}{n}$.

Heuristics: Take K to be a finite complex, projns $\pi_n : K^{\mathbb{Z}} \to K$. Take an open cover $\mathcal{U}_0$ of K with $D(\mathcal{U}_0) = \dim(K)$. Take $\mathcal{U} = \pi_0^{-1}(\mathcal{U}_0) \in \mathrm{Cov}(X)$. Then one gets $D(\mathcal{U} \vee \cdots \vee h^{-n+1}(\mathcal{U})) = n\dim(K)$. (needs work).

One of the main original applications (maybe the main application) was to answer negatively the old & fine question: Let X be cpt metric, let $h: X \to X$ be minimal. Does (X, h) embed in shift on $[0,1]^{\mathbb{Z}}$? No: if $Y \subset X$ is closed inv, then $\mathrm{mdim}(h|_Y) \leq \mathrm{mdim}(h)$. They produce a min homeo h with $\mathrm{mdim}(h) > 1$. Can't embed in shift $h_{[0,1]}$ since it has mdim $= 1$.
h is a subshift of $([0,1]^2)^{\mathbb{Z}}$.

Thm (Downarowicz; culmination of earlier work, starting with the L–W example). Let P be a polyhedron (polyhedral subset of $\mathbb{R}^N$). Let G be a cble amenable gp. Let β satisfy $0 \le \beta < \dim(P)$. Then there is a minimal subsystem of (G, P^G) with $\underline{\dim} = \beta$.
(Both of $\underline{\dim}$ for G is "similar"; use Følner sets in place of $\{0,1,\dots,n-1\} \subset \mathbb{Z}$).

Open problems (1) Can we get a minimal subshift of (G, P^G) with $\underline{\dim} = \dim(P)$?

(2) The construction is complicated, and gives an ugly space:

(2.1) Are there exs of (X,h) with h minimal, X easily described, and $\underline{\dim}(h) > 0$?

(2.2) " " " " h easily described, " " ?

(Really want a "natural example".)

(3) There are other exs of natural non minimal actions with $\underline{\dim} > 0$.
Do they have (interesting, hopefully different) minimal subsystems with $\underline{\dim} > 0$?
(exs: functions with band limited Fourier transforms; Brody curves in several complex variables).

Projections and vector bundles.

A C*-alg. $p \in A$ is **projection** if $p^2 = p$ (idempotent) and $p^* = p$. In $L(H)$: orth proj to subspace.

Defn p and q are Murray–von Neumann equivalent $p \sim q$ (in note file: $p \sim_{MvN} q$) if $\exists s \in A$ s.t. $s^*s = p$ and $ss^* = q$.

In $L(H)$, s is an isometric isom from $\mathrm{Ran}(p)$ to $\mathrm{Ran}(q)$.

In general, "ranges of p, q are isomorphic via an isom in A".

In $L(H)$, $p \sim q$ iff $\dim(\mathrm{Ran}(p)) = \dim(\mathrm{Ran}(q))$ (Hilbert space dimension)

In $C(X)$, proj is χ_K for $K \subset X$ cpt and open. $p \sim q$ iff $p = q$.

Prop $p, q \in A$ projns. TFAE: (1) $p \lesssim q$ in C*-alg order, (2) $pq = p$.
(3) $pq = qp = p$. (4) For some injective repn π of A on H, $\mathrm{Ran}(\pi(p)) \subset \mathrm{Ran}(\pi(q))$.

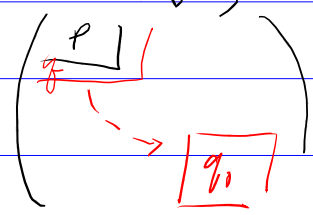
Defn $p \lesssim q$ if $\exists$ projn e st. $p \sim e$ and $e \le q$.

In $L(H)$: $\dim(\mathrm{Ran}(p)) \le \dim(\mathrm{Ran}(q))$; "there is an isom from range of p to subspace of $\mathrm{Ran}(q)$"

To get more out of $C(X)$, etc, go to $M_n(A)$, more generally $M_\infty(A) = \bigcup_n M_n(A)$ infinite matrices in which all but finitely many entries are 0



In $M_\infty(A)$ we can form orthogonal sums of projns.



Get $q_0 \sim q$ but $q_0 \perp p$.
($pq_0 = 0$; in $L(H)$, ranges are orthogonal).

Form $p + q_0$, again a projn, in the direct sum $Ran(p) \oplus Ran(q_0) \subseteq Ran(p) \oplus Ran(q)$.

Easy version to write down: $p \in M_n(A)$, $q \in M_n(A)$.

$p \oplus q \in M_{n+n}(A)$ is $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ as a block matrix.

This gives a well defined commutative semigroup structure on the set $V(A)$ of MvN eq. classes of projns in $M_\infty(A)$. (Exercise: prove this)

Vector bundles (use to produce projns in $M_n(C(X))$.)

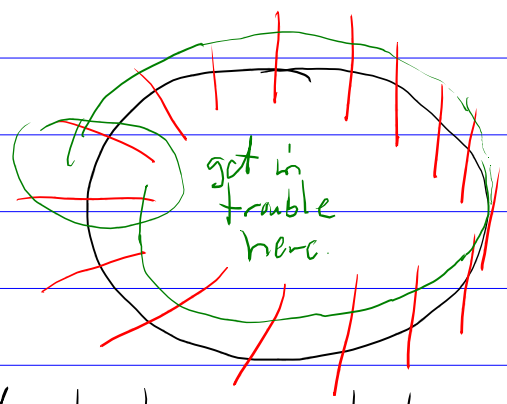
Informally: a vector bundle over X is a family of fin dim $\mathbb{C}$; $\mathbb{R}$ vs sometimes) parametrized by X. Have a top space E, $q: E \to X$, with $E_x = q^{-1}(x)$ a vector space for each $x \in X$.

E_x A vb over a point is a vector space.

E_x Trivial bundle $\mathbb{C}^n \times X$.

E_x M mf, Then TM is a vb (real if M is a real mf.)

E_x Open Möbius strip over its "center line" $X = S^1$, real vb.



Each red line is a copy of $\mathbb{R}$, with 0 where it meets the circle.

A trivial vb has constant sections, nonzero unless use 0 element of the vector space in fact: never zero.

Not trivial. A section of E is a cont. map $s: X \to E$ such that $s(x) \in E_x$ $\forall x \in X$

Try to draw a never zero section of the Möbius bundle. Can't do it; Möbius bundle is nontrivial.

$X = \mathbb{CP}^n$, the set of complex lines in $\mathbb{C}^{n+1}$. E will have fiber E_L over the line L, a pt in $\mathbb{CP}^n$ is L, a one dimensional vector space. "tautological bundle".

$E = \{(\xi, L) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n : \xi \in L\}$, with the relative product topology.

(Need to prove local triviality).

Informal version at first: A vector bundle over X is a family of vector spaces which is locally (on some nbhd U of any $x \in X$) isom to a trivial bundle $\mathbb{C}^{n(x)} \times U$.

There are tautological bundles over Grassmannians.

$E \oplus F$ is done "fiberwise": $(E \oplus F)_x = E_x \oplus F_x$. Get as a subset of $E \times F$ and use product topology.

Swan's Thm: X cpt. Then there is a natural bijection $V(C(X)) \longrightarrow$ isom classes of vbs / X.

Thm sends direct sums of projns $p \oplus q$ as above to direct sum $E \oplus F$ of vbs and $p \precsim q$ iff corresp E, F satisfy E isom to a subbundle of F

subset which is a vb in the relative structure.

(When trying to find subbundles, local triviality is often where the work is.)

(Ex: If $a : E \to F$ is a hom of vbs, then $x \mapsto \ker(a_x)$ may fail to be locally trivial.

Take $E = F = \mathbb{C} \times [0,1]$, $a(\xi, t) = (t\xi, t)$.)

Idea: go via finitely generated projective modules over $C(X)$.

If $p \in M_n(A)$ is a projn, A unital, then p acts on A^n the way one applies a matrix to a vector, and pA^n is a direct summand in A^n.

If E is a vb over X, then space of sections is a $C(X)$-module, fin gen projective when X is compact.

When is E a subbundle of F? (If p, q are corresp projns, when is $p \precsim q$?)

Assume X is connected. $x \mapsto \dim(E_x)$ is locally constant (by local triviality), so constant. Call it $\operatorname{rank}(E)$.

If X is a point then $E \hookrightarrow F$ (E isom to a subbundle of F) iff $\operatorname{rank}(E) \le \operatorname{rank}(F)$.

Existence of nontrivial vbs shows can't happen in general. But things are much worse,

Fact. $G \times X \hookrightarrow E$ iff E has a never zero section. Similarly in real case.

Consider TS^{2n} (tangent bundle).

$n=1$:



If have a never zero section, have a continuous choice of a nonzero tangent vector at each point. This can be seen to be intuitively impossible ("you can't comb the hair on a coconut").

Call it X.

Suppose have a never zero section of TS^{2n}, that is, a never zero vector field. (says smooth.) Use it to make a differential equation: if at $x \in S^{2n}$, find path $t \mapsto y(t)$ such that $y'(t) = X(y(t))$. Fix t: get a cont map $S^{2n} \to S^{2n}$, homotopic to the identity map. Since $X(y)$ is never 0, for small t, $h_t(y) \neq y$. So h has no fixed pts. This contradicts the Lefschetz fixed pt thm. So TS^{2n} is high dim vector bundle, $\mathbb{R} \times S^{2n}$ is small dimensional, but $\mathbb{R} \times S^{2n} \not\hookrightarrow TS^{2n}$.