Recall: $\mathbb{R} \times S^{2n} \not\hookrightarrow TS^{2n}$. This is for real vector bundles. The best thm of this kind for cx vbs: (X connected) $\operatorname{rank}(E) + \frac{1}{2}[\dim(X)-1] \le \operatorname{rank}(F) \Rightarrow E \hookrightarrow F$.

Note: If $X = [0,1]^{\mathbb{Z}}$, then all vbs over X are trivial, so get embedding with $\operatorname{rank}(E) \le \operatorname{rank}(F)$ even with $\dim(X) = \infty$. We will need to look at subspaces of X.

Defn A unital C^*-alg. A **state** ω on A is a lin fcnl $\omega: A \to \mathbb{C}$ such that:

(1) $\|\omega\| = 1$ (2) $\omega(1) = 1$ (3) $a \ge 0 \Rightarrow \omega(a) \ge 0$.

A state τ is a **tracial state** (normalized trace; for shorts, trace) if also $\tau(ba) = \tau(ab)$ $\forall a, b \in A$.

Exs Let Tr_n be the usual trace on M_n. Then $tr_n = \frac{1}{n} Tr_n$ is a tracial state (It is the only one.)

Ex $A = C(X)$. Then the tracial states are $f \mapsto \int_X f d\mu$ for Borel prob measures μ.

Tracial states on $C^*(G, X)$. Assume G stable amenable. For $a \in C^*(G, A)$, there is an associated **formal** series $a \sim \sum_{g \in G} a_g \cdot u_g$. If $G = \mathbb{Z}$ and $A = \mathbb{C}$, then $C^*(G, A) = C(S^1)$ and this series is the Fourier series of a. There is a contractive lin map $E_g: C^*(G, A) \to A$ which gives the g coeff: if $a \sim \sum_n a_n u_n$, then $E_g(a) = a_g$. For $g = 1$, $a \ge 0 \Rightarrow E_1(a) \ge 0$. $\underbrace{\quad}_{E(a)}$.

Recall construction: took faithful repn π_0 of A on H_0, $H = L^2(G, H_0)$ $(u_g \xi)(h) = \xi(g^{-1}h)$, $(\pi(a)\xi)(h) = \pi_0(\alpha_h^{-1}(a))(\xi(h))$. Inside H, have a copy of H_0 for every $g \in G$: $H = \bigoplus_{g \in G} H_0$. Let p be the orth projn to the copy of H_0 at $g = 1$. Then $E(a)$ is determined by $\pi_0(E(a)) = p \underbrace{a}_{\in C^*(G,A) \subset L(H)} p$.

Now look at $A = C(X)$. Let μ be a G-inv. prob measure on X. Define $\tau_\mu(a)$ by $\int_X E(a)\, d\mu$. This is a tracial state. Exercise: verify $\tau_\mu(ab) = \tau_\mu(ba)$, when a, b are in the algebra gen. by A and $\{u_g : g \in G\}$.

Thm If action of G on X is free, then all tracial states on $C^*(G, X)$ are gotten this way.

Notation: $T(A)$ is the set of tracial states on A.

Given $\tau \in T(A)$, extend to $M_n(A)$ by $\tau_n([a_{ij}]) = \sum_{j=1}^{n} \tau(a_{jj})$. Exercise: prove trace property. Note $\tau_n(1) = n$, not 1. Do not renormalize. Instead, extend to $M_\infty(A)$ by, for $a \in M_n(A) \subset M_\infty(A)$, using τ_n. Exercise: This is consistent. Should call result τ_∞, but convention is to use τ.

On $C(X)$, for $a \in M_\infty(C(X))$, one gets $\tau(a) = \int_X Tr_n(a(x))\, d\mu(x)$ if $a \in M_n(C(X))$

Take $A = C(X)$ and look at projections. Assume $\tau \leftrightarrow \mu$. If $p \in M_\infty(C(X))$ has constant rank, then $\mathrm{Tr}(p(x)) = \mathrm{rank}(p)$ $\forall x$, so $\tau(p) = \mathrm{rank}(p)$.

Consider $p, q \in M_\infty(C(X))$, E, F corresp vector bundles. For simplicity take X connected. Then for all $\tau \in T(A)$, $\tau(p) = \mathrm{rank}(E)$ and $\tau(q) = \mathrm{rank}(F)$.

The subbundle thm corresponds to: If $\tau(p) + \frac{1}{2}[\dim(X) - 1] \leq \tau(q)$ $\forall \tau \in T(C(X))$, then $p \lesssim q$. Still true even if X not connected. (exercise).

Want to go to subspaces. Suppose $Y \subset X$ is closed, $p, q \in M_\infty(C(Y))$, q trivial. We know $p \not\lesssim q$. Want to get something on X. Easy to extend q to be a projn $b \in M_\infty(C(X))$. What about p? Extend p to a pos elt $a_0 \in M_\infty(C(X))$. For x near Y, $\frac{1}{2} \notin \mathrm{spl}\, a_0(x)$. ($c \to \mathrm{spl}(c)$ is cont on self-adj elts). Apply fcnl calc with $\chi_{(\frac{1}{2}, \infty)}$ to get $a_1(x)$ for x near Y. $a_1(x)$ is a projn, and x close enough to $Y \Rightarrow \mathrm{rank}(a_1(x)) \leq \sup_{y \in Y} \mathrm{rank}(p(x))$.

$x \in U$.

Get a by $a(x) = f(x) a_1(x)$ with $f : X \to [0,1]$ cont, $= 1$ on Y, supported in U.

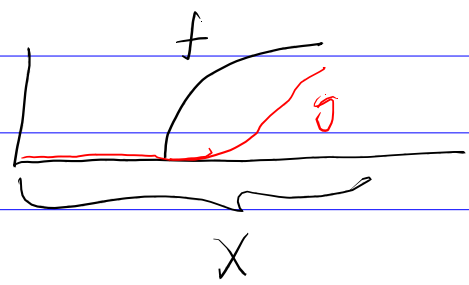
Defn A C*-alg, $a, b \in M_\infty(A)_+$ (B_+ : pos elts in B). We say a is Cuntz subequivalent to b if $\forall \varepsilon > 0$ $\exists$ $v \in M_\infty(A)$ s.t. $\| v^* b v - a \| < \varepsilon$. Write $a \precsim b$.

(For $M_n \lesssim N$ subeq had, if redone: $\| s^* q s - p \| = 0$.)

Using fcnl calc we can show: p, q projns, p Cuntz subeq to $q \Rightarrow p$ $M_n \lesssim N$ subeq to q.

A related statement: p, q projns, $\| p - q \| < 1 \Rightarrow p$ $M_n \lesssim N$ eq to q. (A kind of closeness, not true for analog for Cuntz subeq.)

What is $\lesssim$ in $C(X)$?



Intend: $\{x : g(x) \neq 0\}$
$= \{x : f(x) \neq 0\}$
"open supports" of g, f.

One gets $f \lesssim g$. To see this, look at $(f - \varepsilon)_+ : x \to \max(f(x) - \varepsilon, 0)$.
(by functional calculus makes sense for any pos elt a of a C*-alg. Write $(a - \varepsilon)_+$.)

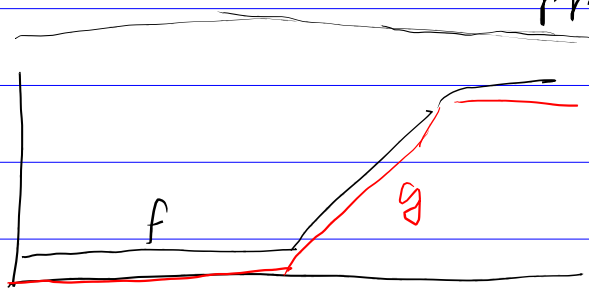
Note: $\overline{\{(f - \varepsilon)_+(x) > 0\}} \subset \{g(x) > 0\}$. Exercise: produce v_ε such that

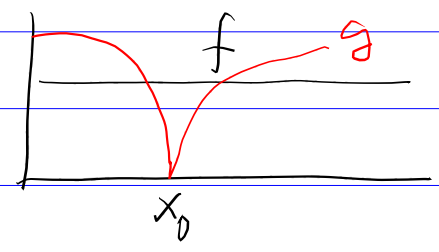
"$\{x \in X :$" is implied.

$v_\varepsilon^* g v_\varepsilon = (f - \varepsilon)_+$. Since $\| (f - \varepsilon)_+ - f \| \leq \varepsilon$, this shows $f \lesssim g$.

In fact, one gets $f \lesssim g$ iff $\{f(x) \neq 0\} \subset \{g(x) \neq 0\}$.

Fails if use usual supports:





$\mathrm{supp}(f) = \mathrm{supp}(g) = X$
but $v^* g v$ vanishes at x_0 $\forall v \in C(X)$.

$\|f-g\| < \varepsilon$ but not $f \sim g$, in fact $f \not\lesssim g$.

$f \lesssim g$ fails badly.

So $f \not\lesssim g$.

One thing that is true: Lemma $\|a-b\| < \varepsilon \Rightarrow (a-\varepsilon)_+ \lesssim b$.

Pf $p \in C(X)$ proj., μ prob. measure. Then $\tau_\mu(p) = \mu(\{x : p(x) = 1\})$
$= \{p(x) \neq 0\}$.

Replacement:
Let A be unital. Let $\tau \in T(A)$, and let $a \in M_\infty(A)_+$. Define $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$ (using functional calculus).

Note: if $\|a\| \leq 1$, then $a^{1/m} \leq a^{1/n}$ for $m \leq n$, so seq is bdd nondecr.
(If $a \in M_\ell(A)$ then $\tau(a^{1/n}) \leq \|a\|^{1/n} \cdot \ell$.)
Check: replacing a by λa with $\lambda > 0$ does not change the limit.
If $f \in (C(X))_+$, $\tau = \tau_\mu$, then $d_\tau(f) = \mu(\{f(x) \neq 0\})$. (check).
$\subset O^*(G,X)$ is exact if G is amenable

Defn Let A be unital and exact. Then A has r-comparison if whenever $a, b \in M_\infty(A)_+$ and $d_\tau(a) + r < d_\tau(b)$ $\forall \tau \in T(A)$, then $a \lesssim b$.
Defn $\mathrm{rc}(A) = \inf(\{r \in [0,\infty) : A \text{ has } r\text{-comparison}\})$. If A is simple and "stably finite", the inf is attained. The defn, applied to $C(X)$, puts vector bundles over subspaces of X in reach.
Conjecture G infinite amenable, X cpt metric, action of G on X minimal and essentially free. Then $\mathrm{rc}(C^*(G,X)) = \frac{1}{2}\dim(G,X)$.
The case $G = \mathbb{Z}$ has become known as the Phillips-Toms Conjecture.
The easier direction seems to be "$\geq$". Now known (Niu) when $G = \mathbb{Z}^d$, and under conditions on action, for somewhat more general G.

Until recently, only known ex with $rc(C^*(G,X)) > 0$ was a subshift, by Gao-Hsu-Kerr in 2018. With Hirshberg, we have a general condition which in particular gives info for the subshifts of Dan.

Special cases. Had $0 \le \beta < \dim(P)$. Write $\beta = \rho \dim(P)$ with $\rho \in [0,1)$.

Ex of what we get: If $P = S^{2n}$, then $rc(C^*(G,X)) \ge \frac{1}{2}\dim(G,X) - 1$.

If $P = [0,1]^{2k+1}$, then $rc(C^*(G,X)) \ge (2\rho - 1)(k-1)$. If $\rho \le \frac{1}{2}$ this is worthless, but if ρ is close to 1, then $(2\rho-1)(k-1)$ is nearly $\dim(G,X) - \frac{3}{2}$.

X is the subshift of P^G. Dan found such a space X that $\dim(G,X) = \rho \dim(P)$. (This was β before.)

Products in cohomology are an essential ingredient in the proof, using Chern character, Chern classes.

Remarks (1) Conjecturally (nearly proved) $rc(A) = 0$ is the right "additional hypothesis" for the Elliott classification program.

(2) Original use was to distinguish counterexs. to Elliott without $rc(A) = 0$ (Toms). The exs. have nothing to do with crossed products.

Cohomology and rc. Elliott-Niu looked at lower bounds on $rc(C(X))$. If X is "good" (covers all finite complexes) [formal condition: $\dim_{\mathbb{Q}}(X) = \dim(X)$], and set $d = \dim(X)$, then

$rc(C(X)) = \max\left(\frac{d-1}{2} - 1, 0\right)$ if d is odd

$= \frac{d}{2} - 1$ or $\frac{d}{2} - 2$ if d is even and these expressions are ≥ 0.

$\dim_{\mathbb{Q}}(X)$ large says X has closed subspaces Y with $H^*(Y;\mathbb{Q}) \ne 0$ in high degrees (roughly $\dim_{\mathbb{Q}}(X)$).

Using Chern character, can get vector bundles from $H^*(Y;\mathbb{Q})$ (need even rational cohomology).

E vb over Y. It has Chern classes $c_j(E) \in H^{2j}(Y)$ for $j \in \mathbb{Z}_{\ge 0}$. Nonzero only when $j \le \operatorname{rank}(E)$.

Properties: $c(E) = \sum_{j=0}^{\operatorname{rank}(E)} c_j(E) \in H^*(Y)$ satisfies $c(E \oplus F) = c(E)c(F)$ (often $c(E) \cup c(F)$).

in ring structure of $H^*(Y)$. The even part is commutative.

$c_0(E) = 1$. If E is trivial, then $c(E) = 1$.

To see this at work. Take $Y = (S^2)^n$, with projns $\pi_k : Y \to S^2$. $S^2 = \mathbb{C}P^1$, so has a tautological bundle E. It has $c(E) = 1+\eta$ for η a generator of $H^2(S^2) \cong \mathbb{Z}$. On Y look at the "pullbacks" $F_k = \pi_k^*(E)$, and $F = \bigoplus_1^n F_k$. Now $H^*(Y)$ is isom to $\mathbb{Z}[\mu_1, \dots, \mu_n] / \langle \mu_j^2 = 0 \ \forall j \rangle$ ($H^*(S^2) \cong \mathbb{Z}[\eta]/\langle \eta^2 \rangle$), and $c(F_k) = 1 + \mu_k$, so $c(F) = \prod_k (1+\mu_k)$.

If $F \hookrightarrow \mathbb{C}^m \times Y$, then $F \oplus F_0 \cong \mathbb{C}^m \times Y$ for some F_0 of rank $m-n$.

So $c(F)\, c(F_0) = 1$. This implies $c(F_0) = \prod_{k=1}^{n} (1-\mu_k)$. It has a nonzero term $\mu_1 \mu_2 \cdots \mu_n$ in $H^{2n}(Y)$. So $\operatorname{rank}(F_0) \geq n$. Therefore $m \geq 2n$.

One can reproduce this (sort of) inside $C^*(\mathbb{Z}, X)$ when $X = (S^2)^{\mathbb{Z}}$, or a subshift with large ρ as in Dow's construction. Further use of ring structure in cohomology.

How? For simplicity assume full shift. on $\underbrace{(S^2)^{\mathbb{Z}}}_{X}$. $A = C^*(\mathbb{Z}, X)$. We have $C(X) \hookrightarrow A$, so $M_\infty(C(X)) \hookrightarrow M_\infty(A)$. Let $p \in M_\infty(C(X))$ correspond to the tautological bundle in the coordinate of X corresponding to $n=0$ in $\mathbb{Z}$.

Inside A, consider finite pieces of orbits, from $-n$ to n (with n large).

$X \to S^2$, start with $p_0 \in M_\infty(C(S^2))$; $C(S^2) \to C(X)$; p is then the image of p_0

Motivation: recall or prd by free action of finite gp G: get matrices of size $\operatorname{card}(G)$. Here get a subalg which looks like $M_{2n+1}(C(S^2))$, in which p looks like it comes from the bundle F above (for $Y = (S^2)^{2n+1}$) (very vague).

If $p \sim$ trivial projn, then this approximately uses only finitely many elts of $\mathbb{Z}$, say in $\{-n_0, \dots, n_0\}$. If $n \gg n_0$, pretend that $n_0 = 0$, and get back to the situation involving Y above. (This is a Følner set at work), nearly.

Current form of Elliott conjecture: Let A and B be simple separable nuclear unital C^*-algs satisfying the Universal Coefficient Theorem, and with one extra condition. Assume that the Elliott invariants of A and B are isomorphic. Then $A \cong B$.

Choices for extra condition: (1) Finite nuclear dimension

(2) $\mathcal{Z}$-stability : ($\mathcal{Z}$ is the Jiang-Su algebra; hyp is $\mathcal{Z} \otimes A \cong A$, $\mathcal{Z} \otimes B \cong B$)

(3) $rc(A) = rc(B) = 0$.

The Toms-Winter conjecture says (1), (2), (3) are equivalent under the other hypotheses of the conjecture, in the stably finite case

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) known. Conj is known assuming (1). (3) $\Rightarrow$ (2) is known in many cases but not full generality. (Should check details!)

Problem Does $\dim(X) = \infty \Rightarrow rc(C(X)) = \infty$?

There are cpt metric spaces X with $\dim(X) = \infty$ but $\dim_{cov}(X) = 3$ (?)
(at least: finite and small).