

Incompressibility for Banach algebras

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A rough outline

- One unrelated question.
- Introduction: Definitions and comments.
- Nonexamples.
- Early examples: C^* -algebras and $L(X)$ for a Banach space X .
- The relation between isometric incompressibility and uniform incompressibility.
- Calkin algebras.
- Incompressibility as a criterion for being “ C^* like”.
- A counterexample.

An unrelated question

This question is not related to the main part of the talk, but arises from the same circle of ideas as some of the motivation.

Let $p \in (1, \infty) \setminus \{2\}$. Let $u \in L(I^p(\mathbb{Z}))$ be the two sided shift operator, $(u\xi)(n) = \xi(n-1)$ for $\xi \in I^p(\mathbb{Z})$ and $n \in \mathbb{Z}$. Let $A \subset L(I^p(\mathbb{Z}))$ be the closed linear span of $\{u^n : n \in \mathbb{Z}\}$, a commutative unital Banach algebra (the group L^p operator algebra of \mathbb{Z}). Does A have spectral synthesis?

The Gelfand transform Γ_A is the continuous extension of the Fourier series map, namely $\Gamma_A: A \rightarrow C(S^1)$ sends $u^n \in A$ to the function $\zeta \mapsto \zeta^n$. (In particular, $\text{Max}(A) \cong S^1$.) Moreover, Γ_A is injective and has dense range.

The question is then whether for every closed ideal I of A there is a closed set $E \subset S^1$ such that

$$I = \{a \in A : \Gamma_A(a) \text{ vanishes on } E\}.$$

If $p = 2$, this is true: $A = C(S^1)$. If $p = 1$, this is known to be false. I have neither found nor been able to prove anything about any other value of p .

Definition of incompressibility

Definition

Let A be a nonzero Banach algebra.

- 1 We say that A is *incompressible* if whenever B is another Banach algebra and $\varphi: A \rightarrow B$ is a bounded injective homomorphism, then φ is bounded below.
- 2 We say that A is *isometrically incompressible* if whenever B is another Banach algebra and $\varphi: A \rightarrow B$ is a contractive injective homomorphism, then φ is isometric.
- 3 We say that A is *uniformly incompressible* if for every $R \in [1, \infty)$ there is $\varepsilon > 0$ such that whenever B is another Banach algebra and $\varphi: A \rightarrow B$ is an injective homomorphism with $\|\varphi\| \leq R$, then for all $a \in A$ we have $\|\varphi(a)\| \geq \varepsilon\|a\|$.

One might want to use the term “incompressible” for what we have called isometric incompressibility. However, this choice leaves no good term for what we have called incompressibility.

Comments on the definition

- 1 A is *incompressible* if every bounded injective homomorphism φ to another Banach algebra is bounded below.
- 2 A is *uniformly incompressible* if in addition the lower bound can be chosen to depend only on $\|\varphi\|$.
- 3 A is *isometrically incompressible* if $\|\varphi\| = 1$ gives the lower bound 1.

In our work, we look a little at the possible ways the lower bound in (2) can depend on $\|\varphi\|$. Suppressed here.

The ideas have a long history, although the names are new, and we have previously unknown relations between them.

For the purposes of the discussion below of early work, we state immediately of our new (and unexpected) results: a Banach algebra A is uniformly incompressible if and only if there is an equivalent norm on A in which A is isometrically incompressible.

C^* -algebras and the algebra of bounded operators on X are isometrically incompressible. (Very early results; below.) We first look at nonexamples.

Triangular matrices are not uniformly incompressible

Example

Let $T \subset L(l^2(\{1, 2\}))$ be the algebra of all upper triangular matrices. Then T is incompressible because it is finite dimensional. However, T is not uniformly incompressible and not isometrically incompressible. To see this, for $\lambda \in \mathbb{C}$ define a homomorphism $\varphi_\lambda: T \rightarrow T$ by

$$\varphi_\lambda \left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ 0 & a_{2,2} \end{pmatrix} \right) = \begin{pmatrix} a_{1,1} & \lambda a_{1,2} \\ 0 & a_{2,2} \end{pmatrix}$$

for $a_{1,1}, a_{1,2}, a_{2,2} \in \mathbb{C}$. One can show (proof omitted) that φ_λ is contractive for all $\lambda \in (0, 1]$. However, there is no uniform lower bound for the homomorphisms φ_λ for $\lambda \in (0, 1]$, since

$$\varphi_\lambda \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}.$$

So T is not uniformly incompressible.

Triangular operators and convolution algebras

The upper triangular matrices in $L(l^2(\{1, 2\}))$ are incompressible but not uniformly incompressible.

The same argument works (with more trickery) for $\{1, 2, \dots, n\}$ in place of $\{1, 2\}$, and (with yet more trickery) shows that the upper triangular elements of $L(l^2(\mathbb{Z}_{>0}))$ are not incompressible.

Also, the disk algebra is not isometrically incompressible, since restriction to the closed disk of radius $\frac{1}{2}$ is injective but not bounded below.

Example

The convolution algebra $l^1(\mathbb{Z})$ is not isometrically incompressible, because the Gelfand transform $\gamma: l^1(\mathbb{Z}) \rightarrow C(S^1)$ is contractive and injective but not bounded below.

These two kinds of examples are essentially the only ways we know to prove that a Banach algebra is not (uniformly) incompressible.

Heuristically, we think of them as “having triangular structure” (this also includes the disk algebra) and “the norm is too large”.

Early examples of incompressibility

Remark

Every finite dimensional Banach algebra is incompressible.

Theorem (Bonsall, 1954)

Let X be a Banach space and let $A \subset L(X)$ be any closed subalgebra that contains all finite rank operators. Then A is isometrically incompressible.

Theorem (Kaplansky, 1949)

Let Ω be a locally compact Hausdorff space. Then every bounded injective homomorphism φ from $C_0(\Omega)$ to another Banach algebra satisfies $\|\varphi(f)\| \geq \|f\|$ for all $f \in C_0(\Omega)$.

In particular, $C_0(\Omega)$ is isometrically incompressible.

Theorem (Bonsall, 1954)

Every C^* -algebra is isometrically incompressible.

The basic incompressibility lemma

Lemma

Let A be a Banach algebra, let $a_0 \in A \setminus \{0\}$ be fixed, and let $C \in (0, \infty)$. Suppose that for all $x \in A$ with $\|x\| = 1$ and all $\varepsilon > 0$, there are $b, c \in A$ such that $\|b\|\|c\| \leq C$ and $\|bxc - a_0\| < \varepsilon$.

- 1 With no further assumptions, A is incompressible.
- 2 Suppose in addition that there is $a_1 \in A \setminus \{0\}$ such that $a_0 a_1 = a_1$. Then A is uniformly incompressible.

In the most common use of (2), a_0 is an idempotent and $a_1 = a_0$.

Sketch of proof of (1).

To simplify, assume we can get $bxc = a_0$. Let $\varphi: A \rightarrow B$ be a bounded injective homomorphism. Let $x \in A$ satisfy $\|x\| = 1$. Choose b and c such that $\|b\|\|c\| \leq C$ and $bxc = a_0$. Then

$$\|\varphi(a_0)\| \leq \|\varphi(b)\|\|\varphi(x)\|\|\varphi(c)\| \leq \|\varphi\|^2 \|b\|\|c\|\|\varphi(x)\| \leq \|\varphi\|^2 C \|\varphi(x)\|.$$

Thus $\|\varphi(x)\| \geq C^{-1} \|\varphi\|^{-2} \|\varphi(a_0)\| = C^{-1} \|\varphi\|^{-2} \|\varphi(a_0)\| \|x\|$. □

The basic incompressibility lemma

Lemma

Let A be a Banach algebra, let $a_0 \in A \setminus \{0\}$ be fixed, and let $C \in (0, \infty)$. Suppose that for all $x \in A$ with $\|x\| = 1$ and all $\varepsilon > 0$, there are $b, c \in A$ such that $\|b\|\|c\| \leq C$ and $\|bxc - a_0\| < \varepsilon$.

- 1 With no further assumptions, A is incompressible.
- 2 Suppose in addition that there is $a_1 \in A \setminus \{0\}$ such that $a_0 a_1 = a_1$. Then A is uniformly incompressible.

We simplified by assuming we can get $bxc = a_0$. Recall: if $\varphi: A \rightarrow B$ is a bounded injective homomorphism, $\|b\|\|c\| \leq C$, and $bxc = a_0$, then (in fact, by scaling, regardless of $\|x\|$), $\|\varphi(x)\| \geq C^{-1} \|\varphi\|^{-2} \|\varphi(a_0)\| \|x\|$.

Sketch of proof of (2).

Now assume in addition that there is $a_1 \in A \setminus \{0\}$ such that $a_0 a_1 = a_1$. Then $\varphi(a_1) \neq 0$, so $\varphi(a_0)\varphi(a_1) = \varphi(a_1)$ implies $\|\varphi(a_0)\| \geq 1$. Thus $\|\varphi(x)\| \geq C^{-1} \|\varphi\|^{-2} \|x\|$. □

The basic incompressibility lemma

Lemma

Let A be a Banach algebra, let $a_0 \in A \setminus \{0\}$ be fixed, and let $C \in (0, \infty)$. Suppose that for all $x \in A$ with $\|x\| = 1$ and all $\varepsilon > 0$, there are $b, c \in A$ such that $\|b\|\|c\| \leq C$ and $\|bxc - a_0\| < \varepsilon$.

- 1 With no further assumptions, A is incompressible.
- 2 Suppose in addition that there is $a_1 \in A \setminus \{0\}$ such that $a_0 a_1 = a_1$. Then A is uniformly incompressible.

There are more complicated versions, which allow a_0 and a_1 to depend on x . (See the proof for $C_0(\Omega)$ below.) But these are almost the only way we know to prove that a Banach algebra satisfies an incompressibility condition.

Incompressibility of $A \subset L(X)$

Theorem

Let X be a Banach space and let $A \subset L(X)$ be any closed subalgebra that contains all finite rank operators. Then A is isometrically incompressible.

Proof.

Write $\langle \omega, \xi \rangle$ for the evaluation pairing $X' \times X \rightarrow \mathbb{C}$. Choose any $\xi_0 \in X$ and $\omega_0 \in X'$ with $\|\xi_0\| = \|\omega_0\| = \langle \omega_0, \xi_0 \rangle = 1$. Define $a_0 \in L(X)$ by $a_0\eta = \langle \omega_0, \eta \rangle \xi_0$ for $\eta \in X$. Then a_0 is a rank 1 idempotent and $\|a_0\| = 1$. Now let $x \in A$ be arbitrary with $\|x\| = 1$. Let $\varepsilon > 0$. Choose $\xi \in X$ such that $\|\xi\| = 1$ and $\|x\xi\| > 1 - \varepsilon$. Choose $\rho \in X'$ such that $\|\rho\| = 1$ and $\langle \rho, x\xi \rangle = \|x\xi\|$. Define rank one operators $b, c \in L(X)$ by

$$b\eta = \langle \rho, \eta \rangle \xi_0 \quad \text{and} \quad c\eta = \langle \omega_0, \eta \rangle \xi$$

for $\eta \in X$. Then $\|b\| = \|c\| = 1$. Also one can easily check that $bxc = \|x\xi\|a_0$. Since $a_0^2 = a_0$, we have verified the hypotheses of part (2) of the basic incompressibility lemma with a_0 as given and $C = 1$. \square

Isometric incompressibility of $C_0(\Omega)$

Theorem (Kaplansky, 1949)

Let Ω be a locally compact Hausdorff space. Then every bounded injective homomorphism φ from $C_0(\Omega)$ to another Banach algebra satisfies $\|\varphi(f)\| \geq \|f\|$ for all $f \in C_0(\Omega)$.

The underlying idea is similar to the basic incompressibility lemma.

Proof.

Let $f \in C_0(\Omega)$ satisfy $\|f\| = 1$ and let $\varepsilon > 0$; we prove that $\|\varphi(f)\| > 1 - \varepsilon$. Replacing f with λf for suitable $\lambda \in \mathbb{C}$, we can assume that there is $x_0 \in \Omega$ such that $f(x_0) = 1$. Choose $g \in C_0(\Omega)$ such that $\|g - f\| < \varepsilon/\|\varphi\|$ and there is a neighborhood U of x_0 such that $g(x) = 1$ for all $x \in U$. Choose a nonzero $h \in C_0(\Omega)$ such that $\text{supp}(h) \subset U$. Then $gh = h$, so $\varphi(g)\varphi(h) = \varphi(h)$. Since $\varphi(h) \neq 0$, this implies $\|\varphi(g)\| \geq 1$. Therefore

$$\|\varphi(f)\| \geq \|\varphi(g)\| - \|\varphi(g) - \varphi(f)\| \geq 1 - \|\varphi\|\|g - f\| > 1 - \varepsilon,$$

as desired. □

Isometric incompressibility of C^* -algebras

Theorem (Bonsall, 1954)

Every C^* -algebra A is uniformly incompressible. In fact, if $\varphi: A \rightarrow B$ is bounded and injective, then $\|\varphi(a)\| \geq \|\varphi\|^{-1}\|a\|$ for all $a \in A$.

In particular, every C^* -algebra is isometrically incompressible. If A is not commutative, the lower bound is the best possible.

Proof.

Let A be a C^* -algebra, let B be a Banach algebra, and let $\varphi: A \rightarrow B$ be a bounded injective homomorphism. Let $a \in A \setminus \{0\}$. Apply the result for $C_0(\Omega)$ to the commutative C^* -algebra generated by a^*a . We get $\|\varphi(a^*a)\| \geq \|a^*a\|$. Therefore

$$\|a\|^2 = \|a^*a\| \leq \|\varphi(a^*a)\| = \|\varphi(a^*)\varphi(a)\| \leq \|\varphi\|\|a^*\|\|\varphi(a)\| = \|\varphi\|\|a\|\|\varphi(a)\|.$$

Divide by $\|\varphi\|\|a\|$. □

Motivation

These results motivated two directions in the theory. The first, with a long history (and where we have some improvements) is to look at incompressibility for Calkin algebras, algebras $L(X)/K(X)$ for a Banach space X . The proof for $L(X)$ depended on the finite rank operators, so fails. Many are not (uniformly) incompressible, but some are.

The second direction, much newer, sees incompressibility of A as an indication that A is “C* like” in some sense. It was motivated by the search for conditions on an L^p operator algebra for it to be “C* like”. (The spectral synthesis problem for the group L^p operator algebra of \mathbb{Z} is a weaker form of this. That algebra isn't incompressible.)

We state carefully the relations between forms of incompressibility. Then we give a brief discussion of incompressibility for Calkin algebras. We give a more extensive discussion of work related to the second direction, since it is more algebraic.

Relations between forms of incompressibility

Recall:

- 1 A is *incompressible* if every bounded injective homomorphism φ to another Banach algebra is bounded below.
- 2 A is *uniformly incompressible* if in addition the lower bound can be chosen to depend only on $\|\varphi\|$.
- 3 A is *isometrically incompressible* if $\|\varphi\| = 1$ gives the lower bound 1.

Here are the easy relations.

Obviously uniform incompressibility implies incompressibility.

The algebra of 2×2 upper triangular matrices shows that incompressibility implies neither uniform incompressibility nor isometric incompressibility.

Proposition

Let $(A, \|\cdot\|)$ be a Banach algebra, and let $\|\cdot\|_0$ be an equivalent algebra norm on A . Then $(A, \|\cdot\|_0)$ is uniformly incompressible if and only if $(A, \|\cdot\|)$ is uniformly incompressible.

Isometric incompressibility and uniform incompressibility

- 1 A is *uniformly incompressible* if a bounded injective homomorphism $\varphi: A \rightarrow B$ has a lower bound depending only on $\|\varphi\|$.
- 2 A is *isometrically incompressible* if $\|\varphi\| = 1$ gives the lower bound 1.

Uniform incompressibility implies incompressibility but the converse is false. Equivalence of norms preserves uniform incompressibility.

The following result is new, and was not suspected. (There are papers on Calkin algebras proving both isometric incompressibility and uniform incompressibility.)

Theorem

Let A be a Banach algebra. Then the following are equivalent.

- 1 A is uniformly incompressible.
- 2 A has an equivalent norm in which it is isometrically incompressible.
- 3 There is $\gamma > 0$ such that whenever B is another Banach algebra and $\varphi: A \rightarrow B$ is a contractive injective homomorphism, then $\|\varphi(a)\| \geq \gamma \|a\|$ for all $a \in A$.

Selected Calkin algebras

If X is a Banach space, we denote by $K(X)$ the closed ideal of all compact operators on X . The Calkin algebra of X is then $Q(X) = L(X)/K(X)$.

Theorem (Meyer 1992)

$Q(X)$ is isometrically incompressible if $X = l^p$ with $p < \infty$, or $X = c_0$,

Theorem (Ware 2014)

Let X be a finite direct sum of spaces in $\{l^p : p \in [1, \infty)\} \cup \{c_0\}$. Then $Q(X)$ is incompressible.

Suppose $1 \leq p_2 < p_1 < \infty$. Then $Q(l^{p_1} \oplus l^{p_2})$ is not uniformly incompressible. The essential reason is that every element of $L(l^{p_1}, l^{p_2})$ is compact, so $Q(l^{p_1} \oplus l^{p_2})$ looks like a “fat” version of the algebra of 2×2 upper triangular matrices.

Theorem (Ware 2014)

Suppose $1 \leq p < \infty$ and $1 \leq r \leq \infty$. Let X be the l^p or c_0 direct sum over $n \in \mathbb{Z}_{>0}$ of the spaces $l^r(\{1, 2, \dots, n\})$. Then $Q(X)$ is incompressible.

More on Calkin algebras

$$Q(X) = L(X)/K(X).$$

$Q(\ell^p)$ is isometrically incompressible for $p \in [1, \infty)$, and $Q(c_0)$ is isometrically incompressible. For finite direct sums of such spaces, one gets incompressibility but generally not uniform incompressibility.

Tarbard (2013) constructed a separable Banach space X such that $Q(X) \cong \ell^1(\mathbb{Z}_{\geq 0})$ with convolution. Thus $Q(X)$ is not incompressible.

Motakis (2024) constructed, for every compact metric space Ω , a separable Banach space X such that $Q(X)$ is isometrically isomorphic to $C(\Omega)$. These are isometrically incompressible.

Ware conjectured that if $p \in (1, \infty) \setminus \{2\}$ then $Q(L^p([0, 1]))$ is incompressible. ($p = 2$ gives a C^* -algebra.) We proved this. In fact, $Q(X)$ is incompressible whenever X is a complemented subspace of $L^p([0, 1])$.

Problem

Is $Q(L^1([0, 1]))$ incompressible?

Is $Q(L^p([0, 1]))$ uniformly incompressible?

Problem

Is $Q(L^p([0, 1]))$ uniformly incompressible?

We assume not. The image S in $Q(L^p([0, 1]))$ of the strictly singular operators is known to be an infinite dimensional closed ideal in which all products are zero.

Proposition

Let A be a Banach algebra and let $J \subset A$ be a closed ideal. Assume:

- 1 J'' is complemented in A'' in the Banach space sense.
- 2 A is left faithful in itself. (If $aA = 0$ then $a = 0$.)
- 3 J is not left faithful in itself. (There is $a \in J \setminus \{0\}$ such that $aJ = 0$.)

Then A is not uniformly incompressible.

Idea: A looks enough like an algebra of upper triangular matrices.

Unfortunately, there is no reason to think that, above, S'' is complemented in $Q(L^p([0, 1]))''$, and in the proposition we don't know how to weaken (1).

Incompressibility and “C* like” Banach algebras

I want a criterion for an algebra in some class of Banach algebras to be a C*-algebra, that makes sense for L^p operator algebras, and that can be used to single out a class of “C* like” L^p operator algebras. The criterion shouldn't implicitly or explicitly use adjoints or selfadjoint elements. (For example, the Vidav-Palmer Theorem assumes the existence of many Hermitian elements. In $L(L^p(X, \mu))$, all Hermitian elements are multiplication operators.)

A test case for a proposed criterion is whether a Hilbert space operator algebra satisfying it is necessarily a C*-algebra.

Among L^p operator algebras, there appear to be several degrees of being “C* like”. For example, group L^p operator algebras are not incompressible, but are probably “weakly C* like”.

The algebra $L(L^p(X, \mu))$ is isometrically incompressible. This, and isometric incompressibility of C*-algebras, motivated the question of whether every isometrically incompressible Hilbert space operator algebra is a C*-algebra.

Technical result: multiplier algebras

A useful technical result.

Proposition

Let A be a Banach algebra that is isometrically faithful in itself, that is, the maps $A \rightarrow L(A)$ sending $a \in A$ to left and right multiplication by a are isometric. Let $M(A)$ be the multiplier algebra of A . Let $D \subset M(A)$ be a closed subalgebra that contains the image of A in $M(A)$. If A is any of incompressible, uniformly incompressible, or isometrically incompressible, then so is D .

“Isometrically faithful in itself” is enough to make sense of $M(A)$ and make $A \rightarrow M(A)$ isometric. This condition holds if A has a contractive approximate identity, but can hold even in the absence of any kind of approximate identity. It does not hold if all products are zero.

The proof isn't hard, but is omitted.

Is every isometrically incompressible Hilbert space operator algebra a C^* -algebra?

The fact that the algebra of 2×2 upper triangular matrices is not isometrically incompressible, and nonincompressibility of the disk algebra, suggest that isometric incompressibility might be a useful criterion for an L^p operator algebra to be “ C^* like”. The following example spoils that idea.

Example

A uniform algebra can be isometrically incompressible but not isomorphic to a C^* algebra. Define subsets $D, \Omega \subset \mathbb{C}$ by $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and $\Omega = \{\zeta \in \mathbb{C} : |\zeta| \leq 2\}$, and set $B = \{f \in C(\Omega) : f|_D \text{ is holomorphic}\}$. Then B is a uniform algebra. (It is a version of the disk algebra with a “thick” boundary.)

By the Maximum Modulus Theorem, the restriction map $B \rightarrow C(\Omega \setminus D) \subset M(C_0(\Omega \setminus \overline{D}))$ is isometric. The multiplier algebra result therefore implies that B is isometrically incompressible.

Quotient isometric incompressibility

$B = \{f \in C(\Omega) : f|_D \text{ is holomorphic}\}$. This is an isometrically incompressible uniform algebra. The disk algebra is a quotient of B , and it is not incompressible. So A is not even isomorphic to a C^* -algebra.

Definition

Let A be a Banach algebra. We say that A is *quotient isometrically incompressible* if whenever B is another Banach algebra and $\varphi: A \rightarrow B$ is a contractive homomorphism, then $\bar{\varphi}: A/\text{Ker}(\varphi) \rightarrow B$ is isometric.

Equivalently, all quotients are isometrically incompressible. C^* -algebras have this property since quotients of C^* -algebras are C^* -algebras.

Known results can be put together to give the following theorem.

Theorem

Let Ω be a locally compact Hausdorff space, and let $A \subset C_0(\Omega)$ be a closed subalgebra. If A is quotient isometrically incompressible, then A is a C^* subalgebra of $C_0(\Omega)$.

(We really only need “quotient incompressibility”.)

Quotient isometric incompressibility and commutative Hilbert space operator algebras

A is *quotient isometrically incompressible* if all quotients are isometrically incompressible.

If $A \subset C_0(\Omega)$ is quotient isometrically incompressible, then A is a C^* -algebra.

Problem

Let H be a Hilbert space, and let $A \subset L(H)$ be commutative and quotient isometrically incompressible. Does it follow that A is a C^* -algebra?

We know that if A contains a nonzero nilpotent element, then A is not uniformly incompressible. Reason: enough triangular structure.

Problem

Let H be a Hilbert space, and let $A \subset L(H)$ be commutative and uniformly incompressible. Does it follow that A contains no nonzero quasinilpotent element?

Quotient isometric incompressibility and operator algebras

If $A \subset L(H)$ is commutative and uniformly incompressible, then A contains no nonzero nilpotent element. Can A contain a nonzero quasinilpotent element?

Problem

Let H be a Hilbert space, and let $a \in L(H)$ be quasinilpotent. Does the Banach algebra it generates contain a nonzero nilpotent element?

I have been told that a positive answer would solve a major case of the Invariant Subspace Problem, but I have not found a counterexample either.

Recall:

Problem

Let H be a Hilbert space, and let $A \subset L(H)$ be commutative and quotient isometrically incompressible. Does it follow that A is a C^* -algebra?

Without commutativity, we have a counterexample. It is much harder than the counterexample for isometric incompressibility.

Orbit breaking subalgebras

Definition

Let Ω be a compact metric space, and let $h: \Omega \rightarrow \Omega$ be a homeomorphism. Let $u \in C^*(\mathbb{Z}, \Omega, h)$ be the standard unitary representing the usual generator of \mathbb{Z} , so that $ufu^* = f \circ h^{-1}$ for $f \in C(\Omega)$. For a closed subset $Y \subset \Omega$, make the identification

$$C_0(\Omega \setminus Y) = \{f \in C(\Omega) : f \text{ vanishes on } Y\},$$

and define the Y -orbit breaking subalgebra $C^*(\mathbb{Z}, \Omega, h)_Y \subset C^*(\mathbb{Z}, \Omega, h)$ by

$$C^*(\mathbb{Z}, \Omega, h)_Y = C^*(C(\Omega), C_0(\Omega \setminus Y)u).$$

Notation

Continuing with the notation above, let $T(\mathbb{Z}, \Omega, h, Y)$ be the norm closed subalgebra of $C^*(\mathbb{Z}, \Omega, h)$ generated by $C^*(\mathbb{Z}, \Omega, h)_Y$ and u .

We do not include u^* among the generators of $T(\mathbb{Z}, \Omega, h, Y)$.

For $x_0 \in \Omega$, abbreviate $C^*(\mathbb{Z}, \Omega, h)_{\{x_0\}} = C^*(\mathbb{Z}, \Omega, h)_{x_0}$ and $T(\mathbb{Z}, \Omega, h, \{x_0\}) = T(\mathbb{Z}, \Omega, h, x_0)$.

$T(\mathbb{Z}, \Omega, h, x_0)$ is quotient isometrically incompressible

$h: \Omega \rightarrow \Omega$ is a homeomorphism. Choose any $x_0 \in X$. Let $u \in C^*(\mathbb{Z}, \Omega, h)$ be the standard unitary. Then $C^*(\mathbb{Z}, \Omega, h)_{x_0} \subset C^*(\mathbb{Z}, \Omega, h)$ is the C^* -algebra generated by $C(\Omega)$ and gu for $g \in C(\Omega)$ with $g(x_0) = 0$, and $T(\mathbb{Z}, \Omega, h, x_0)$ is the Banach algebra gotten by adding to it u but not u^* .

Now assume Ω is infinite and h is minimal. It is not hard to show that $u^* \notin T(\mathbb{Z}, \Omega, h, x_0)$, so $T(\mathbb{Z}, \Omega, h, x_0)$ is not a C^* -algebra. Also, $C^*(\mathbb{Z}, \Omega, h)_{x_0}$ is isometrically incompressible because it is a C^* -algebra. Using this and some work, methods related to the basic incompressibility lemma show $T(\mathbb{Z}, \Omega, h, x_0)$ is isometrically incompressible.

It is known that $C^*(\mathbb{Z}, \Omega, h)$ and $C^*(\mathbb{Z}, \Omega, h)_{x_0}$ are simple.

Theorem

Let B be any Banach algebra with $C^*(\mathbb{Z}, \Omega, h)_{x_0} \subset B \subset C^*(\mathbb{Z}, \Omega, h)$. Then B is simple.

So $T(\mathbb{Z}, \Omega, h, x_0)$ is isometrically incompressible and simple, thus quotient isometrically incompressible.

Further problems

Incompressibility nevertheless seems to possibly be a property that “strongly C^* like” L^p operator algebras should have.

Let Ω be compact metric, and let h be a minimal homeomorphism of Ω . I think I know how to prove that the L^p operator crossed product $F^p(\mathbb{Z}, \Omega, h)$ is isometrically incompressible, but the intended proof is long.

Problem

Are the L^p Cuntz algebras isometrically incompressible?

They are purely infinite and simple, and “isometrically incompressible among L^p operator algebras”. But an example of Daws and Horváth shows that pure infiniteness does not imply incompressibility.

Problem

If $p \in (1, \infty) \setminus \{2\}$, is the reduced L^p group operator algebra of F_2 isometrically incompressible?

These algebras are simple. (For $p = 1$, this algebra is neither simple nor incompressible.)