Math 635: Algebraic Topology II, Winter 2015
Homework #6: Mayer-Vietoris and universal coefficients
Exercises from Hatcher: Chapter 2.2, Problems 28, 29, 30, 32, 33, 40.

28a. Let $X = T \cup M$, the union of a torus and a Möbius band along a circle. Our space X is a 2-dimensional cell complex, so its homology vanishes in degree greater than 2. The map from $H_1(S^1) \cong \mathbb{Z}$ to $H_1(T) \oplus H_1(M) \cong \mathbb{Z}^3$ takes a to $(a, 0, 2a)$. In particular, it is injective, which means that the map from $H_2(X)$ to $H_1(S^1)$ is zero, and we have

$$0 \to H_2(T) \oplus H_2(M) \to H_2(X) \to 0.$$

Thus $H_2(M) \cong H_2(T) \oplus H_2(M) \cong \mathbb{Z}$. Similarly, the map from $H_0(S^1)$ to $H_0(T) \oplus H_0(M)$ is injective, so the map from $H_1(X)$ to $H_0(S^1)$ is zero, and we have

$$0 \to H_1(S^1) \to H_1(T) \oplus H_1(M) \to H_1(X) \to 0.$$

Thus $H_1(X)$ is isomorphic to the cokernel of the aforementioned map, which is isomorphic to $\mathbb{Z}^2$. Since X is path connected, $H_0(X) \cong \mathbb{Z}$.

28b. Let $Y = \mathbb{R}P^2 \cup M$. As in part (a), the homology of Y vanishes in degree greater than 2, and the degree 0 homology is isomorphic to $\mathbb{Z}$, so we only need to look at degrees 1 and 2. The map from $H_1(S^1) \cong \mathbb{Z}$ to $H_1(\mathbb{R}P^2) \oplus H_1(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ takes a to $(\bar{a}, 2a)$. This is injective, so $H_2(X) \cong H_2(\mathbb{R}P^2) \oplus H_2(M) = 0$. As above, $H_1(X)$ is isomorphic to the cokernel of this map, which is isomorphic to $\mathbb{Z}_4$. (It is the quotient of $\mathbb{Z}^2$ by the subgroup generated by $(2, 0)$ and $(1, 2)$. Looking at the determinant tells us that it has order 4, and the element $(1, 1)$ has order 4, so the group is cyclic.)

29. The space R is homotopic to a wedge of g circles, so $H_1(R) \cong \mathbb{Z}^g$, $H_0(R) \cong \mathbb{Z}$, and the rest of the homology groups of R vanish. This tells us that $H_3(X) \cong H_2(M_g) \cong \mathbb{Z}$. It also tells us that $H_2(X)$ is isomorphic to the kernel of the map from $H_1(M_g) \cong \mathbb{Z}^{2g}$ to $H_1(R) \oplus H_1(R) \cong \mathbb{Z}^{2g}$, and $H_1(X)$ is isomorphic to the cokernel of the same map. The map is given by projecting from $\mathbb{Z}^{2g}$ onto $\mathbb{Z}^g$ and then including diagonally into $\mathbb{Z}^{2g}$, which means that the kernel and cokernel are each isomorphic to $\mathbb{Z}^g$.

30. In each case, only H_1, H_2, and H_3 are interesting. Let $\psi_i = \mathrm{id} - f_* : H_i(X) \to H_i(X)$, and note that $\psi_0 = 0$. We have $H_3(T_f) \cong \ker(\psi_2)$,

$$0 \to \mathrm{coker}(\psi_2) \to H_2(T_f) \to \ker(\psi_1) \to 0,$$

and

$$0 \to \mathrm{coker}(\psi_1) \to H_1(T_f) \to H_0(X) \to 0.$$

In our examples, $H_1(X)$ has no torsion, so $\ker(\psi_1)$ has no torsion, so the first sequence always splits. Similarly, $H_0(X)$ has no torsion, so the second sequence also splits. So if we can compute the kernel and cokernel of ψ_2 and ψ_1, then we know everything.

a) $H_1(S^2) = 0$, and $H_2(S^2) \cong \mathbb{Z}$, with ψ_2 equal to multiplication by 2. Thus $\ker(\psi_2) = 0$ and $\mathrm{coker}(\psi_2) \cong \mathbb{Z}_2$.

b) ψ_2 is multiplication by -1, so its kernel and cokernel are trivial.

c) $\psi_1(a,b) = (0, 2b)$, so $\ker(\psi_1) \cong \mathbb{Z}$ and $\operatorname{coker}(\psi_1) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. $\psi_2(c) = 2c$, so $\ker(\psi_2) = 0$ and $\operatorname{coker}(\psi_2) \cong \mathbb{Z}_2$.

d) $\psi_1(a,b) = (2a, 2b)$, so $\ker(\psi_1) = 0$ and $\operatorname{coker}(\psi_1) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $\psi_2 = 0$, so $\ker(\psi_2) \cong \mathbb{Z} \cong \operatorname{coker}(\psi_2)$.

e) $\psi_1(a,b) = (a+b, b-a)$, so $\ker(\psi_1) = 0$ and $\operatorname{coker}(\psi_1) \cong \mathbb{Z}_2$. $\psi_2 = 0$, so $\ker(\psi_2) \cong \mathbb{Z} \cong \operatorname{coker}(\psi_2)$.

32. This follows from the Mayer-Vietoris sequence for $SX = CX \cup CX$, using the fact that CX is contractible.

33. We'll prove the statement by induction on n. When $n = 1$, it is clear. Now assume that it holds for $n-1$, and let $Y = A_1 \cup \cdots \cup A_{n-1}$. Our inductive hypothesis, applied to both Y and $Y \cap A_n$, tells us that $\tilde{H}_i(Y) = 0 = \tilde{H}_i(Y \cap A_n)$ for all $i \geq n-2$. The Mayer-Vietoris for $X = Y \cup A_n$ gives us

$$\tilde{H}_i(Y) \oplus \tilde{H}_i(A_n) \to \tilde{H}_i(X) \to \tilde{H}_{i-1}(Y \cap A_n).$$

If $i \geq n-1$, then $i - 1 \geq n-2$, and we win.

40. The long exact sequence gives us

$$H_i(X) \xrightarrow{n} H_i(X) \to H_i(X; \mathbb{Z}_n) \to H_{i-1}(X) \xrightarrow{n} H_{i-1}(X).$$

Replacing the first map with the its cokernel and the last map with its kernel, we obtain the desired short exact sequence. We can also put tildes on all of these groups for free. (When $i > 1$, it changes nothing. When $i = 1$, it also changes nothing, since $H_0(X)$ has no torsion. When $i = 0$, it is clearly fine.)

Then $\tilde{H}_i(X, \mathbb{Z}_p) = 0$ if and only if $\tilde{H}_{i-1}(X)$ has no p-torsion and the operation of multiplication by p on $\tilde{H}_i(X)$ is invertible. This is true for all p if and only if $\tilde{H}_{i-1}(X)$ is torsion-free and $\tilde{H}_i(X)$ is a $\mathbb{Q}$-vector space. It is true for all p and all i if and only if $\tilde{H}_i(X)$ is a $\mathbb{Q}$-vector space for all i.

Note: The necessity for passing to reduced groups is a little bit funny. The statement is also true for nonreduced homology groups, but it is (nearly) vacuous: $H_0(X; \mathbb{Z}_p)$ is never zero (unless X is empty), and $H_0(X)$ is never a $\mathbb{Q}$-vector space (unless X is empty).